p-adic L functions for totally real number fields I

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§ Introduction

F totally real number field, $G_F = \mathrm{Gal}(\bar{F}/F)$.

$\psi: G_F \to \mathbb{C}^\times$ continuous: Artin L function of ψ is

(meromorphic continuation to all $\mathbb{C}$ of $s \mapsto \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}_F} \frac{\psi((\mathfrak{a}, F^{ab}/F))}{N\mathfrak{a}^s}$, $Re(s)>1$ where $(\mathfrak{a}, G^{ab}/G)$ is the Artin symbol of CFT)

The aim of my series of talks is to make precise what we mean by a "p-adic analogue of this complex function".

1 (today). Definitions & strategy for construction

2. Deligne-Ribet (1978) construction

3. Cassou-Noguès (1979) & Barsky (1977) construction.

§ Definitions

The totally real number field F is fixed from now on.

- $p>2$ an odd prime, $F_\infty = F\cdot\mathbb{Q}_\infty$ cyclotomic ext, $\Gamma = \mathrm{Gal}(F_\infty/F)$
- M/F = finite abelian real ext, $F_\infty \cap M = F$. Set also $M_\infty = MF_\infty$ and $G = \mathrm{Gal}(M_\infty/F) \xrightarrow{\kappa_{cyc}} \Gamma \xrightarrow{\sim} 1+p\mathbb{Z}_p$. $\Delta := \mathrm{Gal}(M/F)$

Def Let $\varepsilon: G \to \bar{\mathbb{Q}}$ be locally constant. Then set

$$L(\varepsilon, s) = \sum_{\substack{0 \neq \mathfrak{a} \subseteq \mathcal{O}_F \\ \mathfrak{a} \text{ unramified in } M_\infty/F}} \varepsilon(\mathfrak{a}) N\mathfrak{a}^{-s} \quad \text{for } Re(s)>1 \quad \left[\varepsilon(\mathfrak{a}) := \varepsilon((\mathfrak{a}, M_\infty/F))\right]$$

$\Rightarrow$ It depends on G! E.g. $\mathbb{1}: G \to \mathbb{Q}$, $g \mapsto 1$ is defined for all G

but $L(\mathbb{1},s)$ depends on it!

Thm (Siegel) (i) $L(\varepsilon,s)$ extends meromorphically to $\mathbb{C}$, only possible pole is simple at $s=1$.

(ii) $\forall\, j\leq 0$, $L(\varepsilon,j)\in\mathbb{Q}(\varepsilon)$

$\Longrightarrow$ If V is a $\mathbb{Q}$-vector space, $\varepsilon: G\to V$ is locally const. (e.g. character $\neq \pm 1 \Rightarrow V=\mathbb{Q}(\varepsilon)\supsetneq\mathbb{Q}$), $L(\varepsilon,j): G\to\mathbb{C}$ well defined

Pf $N<G$ s.t. $\varepsilon(x)$ depends only on $x \pmod N$: it is open $\Rightarrow$ finite index $\Rightarrow$ set $V_{\bar{x}}=\varepsilon(\bar{x})$ for $\bar{x}\in G/N$: then put

$$L(\varepsilon,j)=\sum_{\bar{x}\in G/N} L(\chi_{\bar{x}+N},j)\cdot V_{\bar{x}} \quad \text{where } \chi_*: G\to\mathbb{Q} = \text{char. fun.}$$ $\square$

Naive Def 1: Let $\psi:\Delta\to\overline{\mathbb{Q}}_p^\times$ be a char. ($\Rightarrow \psi: G\twoheadrightarrow\Delta\to\overline{\mathbb{Q}}_p^\times$ loc. const, $V=\mathbb{Q}_p(\psi)$). Then the p-adic L-function "could be" the continuous function $L_p(\psi,s):\mathbb{Z}_p\to\mathbb{C}_p$ s.t. $L_p(\psi,j)=L(\psi,j)$, $\forall j\leq 0$.

$\Rightarrow$ Does not exist! $\psi(-1)=-1\Rightarrow L(\psi,j)=0$ $\forall\, 0\geq j\equiv 0\ (2)\Rightarrow L_p(\psi,j)\equiv 0$

$\Rightarrow L_p(\psi,j)=0\neq L(\psi,j)$ if $0\geq j\equiv 1\ (2)$

Naive Def. 2: $L_p(\psi,j)=L(\psi\omega^{1-j},j)$ $\forall\, j\leq 0$ $\left((-1)^{j-1}=\omega^{j-1}(-1)\cdot\underset{1}{\underset{\shortparallel}{\eta(-1)}}\right)$

Rmk 1. Does it work? $\to$ Almost.

2. Why should we (= you) care? $\to$ As in Ochiai's talk, $L_p(\psi,-)\leftrightarrow$ Iwasawa Theory!

Now we proceed to a more precise definition.

Thm Let $(V,|\cdot|)$ be a normed $\mathbb{Q}$-vector space. Then

$$LC(G,V)=\{\varepsilon: G\to V \text{ locally const.}\}\underset{\text{dense!}}{\subset}\mathcal{C}(G,V)=\{f:G\to V \text{ cts}\}$$

Rmk $G=$ any commutative profinite gp would be ok.

Pf $f: G \to V$ cts $\Rightarrow$ bounded (G is compact) & uniformly cts (G is Hausdorff). So

- $\|f\| = \sup_{g \in G} |f(g)|$ is a norm
- $\forall\ \delta > 0\ \exists\ H = H(\delta) \trianglelefteq G$ s.t. $g \cdot g' \in H \Rightarrow |f(g) - f(g')| < \delta$.

$\Rightarrow$ Look for $\varepsilon \in LC(G,V)$ s.t. $\|f - \varepsilon\| < \delta$.

Choose representatives $g_1, \ldots, g_s$ of $G/H(\delta)$, set $\varepsilon|_{g_i + H} = f(g_i)$, loc. const. Then $\sup_{g \in G} |f(g) - \varepsilon(g)| = \sup_i \sup_{g \in g_i + H} |f(g) - \varepsilon(g)|$

$$= \sup_{1 \le i \le s} \sup_{g} |f(g) - f(g_i)| < \delta$$

$\square$

Def A V-valued measure is a linear functional $\mu: \mathcal{C}(G,V) \to V$ such that $|\mu(f)| \le \|f\|\ \ \forall\ f \in \mathcal{C}(G,V)$.

We say μ is rational if for every subspace $W \subseteq V$, and all f, we have $f(G) \subseteq W \Rightarrow \mu(f) \in W$.

Rmk Sometimes it is only required $\exists\ c$ s.t. $|\mu(f)| \le c \cdot \|f\|$.

Cor: V complete $\Rightarrow$ Measure determined by its values on $LC(G,V)$. If V is a field, it is uniquely determined by $\mathrm{Hom}_{cont}(G, V^*)^{f.o.}$, the characters of G of finite order.

Pf. Bounded $\Rightarrow$ cts; and V Hausdorff $\Rightarrow$ cts fct is defined on dense subspace. By linear indep. of characters, every $\varepsilon \in LC(G,V)$ is uniquely finite, linear comb. of characters

$\square$

Recall $\Delta = \mathrm{Gal}(M/F)$: then $G = \Delta \times \Gamma \Rightarrow G \ni g \overset{!}{=} (\delta, \gamma)$

$\rightsquigarrow \psi \in \mathrm{Hom}(\Delta, \overline{\mathbb{Q}}_p^\times) \Rightarrow$ Induced map $(\pi^\psi)^*: \mathcal{C}(\Gamma, \mathbb{C}_p) \to \mathcal{C}(G, \mathbb{C}_p)$

via $((\pi^{\psi})^{*}f)(\delta,\gamma)=\psi(\delta)f(\gamma) \Rightarrow \pi^{(\psi)}_{*}: \mathcal{M}(G,\mathbb{C}_p) \longrightarrow \mathcal{M}(\Gamma,\mathbb{C}_p)$

$$\mu \longmapsto (\pi^{\psi}_{*}\mu)(f) := \mu((\pi^{\psi})^{*}f)$$

It preserves rationality if $\psi(\Delta)\subseteq V$.

Def (i) If $\psi\neq 1$, the p-adic L-function attached to ψ is the unique rational $\lambda_{\psi}\in\mathcal{M}(\Gamma,\mathbb{C}_p)$ s.t. $\forall j\leq 0$

$$\int \phi\cdot \kappa_{cyc}^{j}\, d\lambda_{\psi} = L(\phi\cdot\psi, j) \quad \forall\, \phi:\Gamma\to\overline{\mathbb{Q}}_p^{\times} \text{ of finite ord,}$$

(ii) If $\psi=1$, fix an ideal $\mathfrak{c}\subseteq \mathcal{O}_F$, unramified in M_{∞}/F. The p-adic zeta function with parameter $\mathfrak{c}$ is the unique rational $\zeta_{p,\mathfrak{c}}\in\mathcal{M}(\Gamma,\mathbb{C}_p)$ s.t.

$$\int \phi\cdot\kappa_{cyc}^{j}\, d\zeta_{p,\mathfrak{c}} = \left(1-(N\mathfrak{c})^{1-j}\cdot\phi(\mathfrak{c})\right)L(\phi,j) \quad \forall\,\phi, j\leq 0$$

Rmk · Uniqueness, if they exists, is clear by Corollary.

· Why only measures on Γ and not on G?

· Why, if $\psi=1$, the stupid factor "$1-(N\mathfrak{c})^{1-j}\cdot\phi(\mathfrak{c})$"?

$\Rightarrow$ The original construction produces a linear functional $\lambda:\mathcal{C}(G,\mathbb{C}_p)\to\mathbb{C}_p$ which, unfortunately, is not bounded. But it works for all ψ at the same time (after all, if $\varepsilon\in LC(G,\mathbb{C}_p)$, $\psi\cdot\varepsilon$ is also locally constant). In other words, $\lambda_{\psi}=\pi^{(\psi)}_{*}\lambda \rightsquigarrow$ Of course, such a λ seems a nicer object: but, for Iwasawa-theoretic purposes - is "less" suited.

If $\psi=1$, the "stupid factor" helps keeping it bounded. Indeed,

(i) construct $\mu_{\mathfrak{c}}\in\mathcal{M}(G,\mathbb{C}_p)$ s.t. $\int \kappa_{cyc}^{j}\,\varepsilon\, d\mu_{\mathfrak{c}} = (1-\kappa_{cyc}^{1-j}(\mathfrak{c})\varepsilon(\mathfrak{c}))L(\varepsilon,j)$

(ii) "divide" μ_c by $(1-\delta_c)$, so $\lambda = \frac{\mu_c}{(1-\delta_c)} \in \mathrm{Frac}(\mathcal{M}(G, \mathbb{C}_p))$

(iii-1) $\psi \neq 1 \Rightarrow$ Check that $\pi_*^\psi \lambda$ is bounded again.

(iii-2) $\psi = 1 \Rightarrow \pi_*^\psi \lambda$ not bounded $\Rightarrow$ just set $\zeta_{p,c} = \pi_*^\psi(\mu_c)$.

- A priori, $\forall j \leq 0$ the functional $\varepsilon \mapsto L(j, \varepsilon\psi)$ (if $\psi \neq 1$) is a measure $\lambda_\psi^{(j)}$. Not clear:
  - What is $\int \kappa_{cyc}^t \cdot \varepsilon \, d\lambda_\psi^{(j)}$
  - Relation $\lambda_\psi^{(j)} \longleftrightarrow \lambda_\psi^{(j')}$.

  Thm (proven below): $\forall j \leq 0$, $\lambda_\psi^{(j)} = \kappa_{cyc}^j \cdot \lambda_\psi^{(0)}$. Then set $\lambda_\psi = \lambda_\psi^{(0)}$ and both questions are settled.

§ Relations with Iwasawa theory, Ochiai's lecture & "Dream"

- In this §, assume $\lambda_\psi, \zeta_{p,c}$ exist!

Prop There is an isomorphism:

$$\Lambda_\psi \xrightarrow[\sim]{\mathcal{F}} \mathcal{M}(\Gamma, \mathbb{Q}_p(\psi))^{\mathrm{rat.}}$$
$$\downarrow \Gamma_p$$
$$\mathcal{C}(\mathbb{Z}_p, \mathbb{C}_p)$$

, where

- $\mathcal{F}\left(\sum a_n (\gamma-1)^n\right)(\varepsilon) = \sum_{i=0}^{p^N-1} \varepsilon(\gamma^i) \cdot a_i$ if ε constant (mod Γ^{p^N})
- $\Gamma_p(\mu)(s) = \int_\Gamma \left(\kappa_{cyc}^{-1}(x)\right)^s d\mu$ ("p-adic Γ-transform")

$\Rightarrow$ Ochiai's $L_p(\psi) = \mathcal{F}^{-1}(\lambda_\psi)$

Naive $L_p(\psi, s) = \Gamma_p(\lambda_\psi)$

$*$ $\psi = 1$? λ_ψ does not exist... $\Rightarrow \mathcal{F}^{-1}(\zeta_{p,c}) \underset{\mathcal{F}}{=} (1-\gamma) L_p(\psi)$

If $(c, M_\infty/F) = \gamma$

§ Strategy for constructing λ_ψ

Theorem (Coates) Assume that

[A] For all $\varepsilon \in LC(\Gamma, \mathbb{Z}_p)$, $\forall$ c unramified in M_∞/F,

$$L(0,\psi\varepsilon) - N(c)\psi(c) L(0,\psi\varepsilon_c) \in \mathbb{Z}_p(\psi)$$

[B] For all $j \leq 0$ and $n \geq 0$, let $LC(\Gamma, \mathbb{Z}_p) \ni \eta \equiv k_{cyc}^{-j}$ modulo p^n. Then $\forall$ $\varepsilon \in LC(\Gamma, \mathbb{Z}_p)$

$$L(j,\psi\varepsilon) - (Nc)^{1-j}\psi(c) L(j,\psi\varepsilon_c) \equiv L(0,\psi\varepsilon\eta) - N(c)\psi(c) L(0,\psi\varepsilon_c\eta_c) \quad \left(\varpi^n_{\mathbb{Z}_p(\psi)}\right)$$

Then, if $\psi \neq 1$, λ_ψ exists; if $\psi = 1$ $\zeta_{p,c}$ exists.

Here we have set $\varepsilon_c : g \mapsto \varepsilon(c; M_\infty/F)\cdot g)$

Pf Let $\mu_c : LC(G, \overline{\mathbb{Q}}_p) \longrightarrow \overline{\mathbb{Q}}_p$ be defined by [A]

By assumption, it is bounded $\Rightarrow \mu_c \in \mathcal{M}(G, \overline{\mathbb{Q}}_p)$. Look at $\frac{\mu_c}{1-\delta_c} \in \mathrm{Frac}(\mathcal{M}(G, \overline{\mathbb{Q}}_p))$. It can easily be checked to be a functional, independent of c. By using $\mathcal{F}$ above $\Rightarrow \frac{\mu_c}{1-\delta_c} = \lambda + \eta$ with $\lambda \in \mathcal{M}(G, \overline{\mathbb{Q}}_p)$, η induced from a Haar measure on Δ. Then $(\pi^\psi_*)(\eta) = \begin{cases} 0 & \psi \neq 1 \\ \neq 0 & \psi = 1 \end{cases}$

- If $\psi \neq 1$, $\pi^\psi_*(\lambda+\eta) := \lambda_\psi \in M(\Gamma, \overline{\mathbb{Q}}_p)$ independent of c.
- If $\psi = 1$, no! Just push-forward μ_c

Now we have to use [B] to check what happens if we integrate $k_{cyc}^j \cdot \varepsilon$ against λ_ψ. For simplicity, only $\psi \neq 1$:

ETS: $\int k_{cyc}^j \cdot \varepsilon \, d\lambda_\psi \equiv L(j,\varepsilon) \quad \varpi^n := \varpi^n_{\mathbb{Z}_p(\psi)} \quad \forall n \geq 0$

Pick η as in [B]: then (μ_c linear)

$$L(j,\psi\varepsilon) - (Nc)^{1-j}\psi(c) L(j,\psi\varepsilon_c) \overset{[B]}{\equiv} \int \varepsilon\eta \, d\mu_c \equiv \int \varepsilon k_{cyc}^j \, d\mu_c \quad (\varpi^n)$$

$\Rightarrow$ Apply $\pi_x^{(\chi)}$ to get the desired congruences $\square$