p-adic L functions for totally real number fields II

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§ Introduction

Recall: $F \neq \mathbb{Q}$ tot. real, $p>2$, M/F abelian, $\Delta = G(M/F)$
M real, $M \cap F_\infty = F$ and $G = \mathrm{Gal}(M_\infty/F)$; $\psi : \Delta \to \overline{\mathbb{Q}}_p^\times$.
We write $\mathfrak{d} = \mathrm{diff}(F/\mathbb{Q})^{-1} \subseteq \mathcal{O}_F$

End of yesterday lecture:

[A] For all $\varepsilon \in LC(\Gamma, \mathbb{Z}_p)$, $\forall$ $\mathfrak{c}$ unramified in M_∞/F,
$L(0,\psi\varepsilon) - (N\mathfrak{c})\,\psi(\mathfrak{c})\,L(0,\psi\varepsilon_{\mathfrak{c}}) \in \mathbb{Z}_p(\psi)$

[B] For all $j \le 0$ and $n \ge 0$, let $LC(\Gamma, \mathbb{Z}_p) \ni \eta \equiv \kappa_{cyc}^{-j}$ modulo p^n. Then $\forall$ $\varepsilon \in LC(\Gamma, \mathbb{Z}_p)$

$$L(j,\psi\varepsilon) - (N\mathfrak{c})^{1-j}\psi(\mathfrak{c})L(j,\psi\varepsilon_{\mathfrak{c}}) \equiv L(0,\psi\varepsilon\eta) - (N\mathfrak{c})\,\psi(\mathfrak{c})\,L(0,\psi\varepsilon_{\mathfrak{c}}\eta_{\mathfrak{c}}) \quad (\varpi^n).$$

Look at the following power series, for $\varepsilon \in LC(G, \mathcal{O}_p)$ & $\mathfrak{a} \subseteq \mathcal{O}_F$

$$F(\mathfrak{a}) = N(\mathfrak{a})\psi(\mathfrak{a})\Big\{L(0,\varepsilon_{\mathfrak{a}}\psi) + \sum_{\mu \in ?} \varepsilon_{\mathfrak{a}}(\mu)\,q^\mu\Big\} \in \mathbb{Q}_p(\psi)[[q]].$$

$\Rightarrow$ (i) $F(1) - F(\mathfrak{c})$ has equation in [A] as constant term.
(ii) If $\varepsilon \in LC(\Gamma, \mathbb{Z}_p) \Rightarrow$ all non-constant terms in $\mathbb{Z}_p$.

<u>Main Idea</u>: Find a "modular form" $\mathcal{F}$ on some space s.t. $F(\mathfrak{a})$ is the "q-expansion of $\mathcal{F}$ at a certain cusp", $\forall$ $\mathfrak{a}$
$\Rightarrow$ "Thm (DR)": If $\mathcal{F}$ is such a "modular form" and all non-constant coeff. of the q-exp. at one cusp are integral, then the difference between constant terms at

different cusps is also integral.

$\Rightarrow$ [A] follows. For [B], same trick, but replace F with a modular form containing L values both for ε and η. We only treat [A], [B] being analogous.

§ Hilbert–Blumenthal Abelian Varieties

$\mathrm{Spec}(R)$ = affine scheme (for simplicity), $X \xrightarrow{p} \mathrm{Spec}(R)$ abelian scheme (= proper, smooth, R-gp. scheme with geom. connected fibers).

$\underline{\omega}_{X/R} := p_* \Omega^1_{X/R}$ the sheaf of invariant differentials
$= R$-dual to $\mathrm{Lie}(X/R)$, sheaf of invariant vector fields.
$=$ locally free R-module of rank $\dim_R(X)$.

Def A (F-) Hilbert-Blumenthal Abelian Variety (HBAV) is an abelian scheme $X \xrightarrow{p} \mathrm{Spec}(R)$ with a structure $\mathcal{O}_F \hookrightarrow \mathrm{End}_R(X)$ making $\underline{\omega}_{X/R}$ (or $\mathrm{Lie}(X/R)$) into a locally free $\mathcal{O}_F \otimes_{\mathbb{Z}} R$-module of rank 1.

Rmk
- X/R HBAV $\Rightarrow \dim_R X = [F:\mathbb{Q}]$
- ω an $\mathcal{O}_F \otimes R$-basis of $\underline{\omega}_{X/R} \Longleftrightarrow \mathcal{O}_F \otimes R$-iso $\mathrm{Lie}(X/R) \xrightarrow{\sim} \partial \otimes R$
- E/R an elliptic curve with CM by $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})} \Longleftrightarrow \mathrm{End}(E)$ is $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ free of rank 1. This is the "real analogue".
  $\Rightarrow$ HBAV are sometimes referred to as "abelian varieties with real multiplication by F".
- $\dim_R X = 1$ (i.e. ell. curve) $\Rightarrow F = \mathbb{Q}$: only "real mult" for e.c. is $\mathbb{Z}$...

· F and R have <u>nothing</u> to do with each other. So, we speak about F-HBAV over R, or R-HBAV (understood: via F) if F is fixed. $\Rightarrow$ For us, it is fixed henceforth!

$\rightsquigarrow$ Let $X/_R$ be a HBAV, $N \geq 1$. A $\Gamma_{00}(N)$-structure on X is a embedding $\partial \otimes_{\mathbb{Z}} \mu_N \hookrightarrow X$ as $\mathcal{O}_F \otimes R$-gp. schemes.

Def 1) A modular form $\mathcal{F}$ of level $\Gamma_{00}(N)$ and weight $k \in \mathbb{Z}/_R$ is a rule $(X/_S, \lambda, \omega, i) \xrightarrow{\mathcal{F}} S$

where $*$ $X/_S$ = HBAV over S = R-algebra
$*$ $\lambda : X^t \xrightarrow{\sim} X$ principal polariz.
$*$ $\omega : \mathrm{Lie}(X/_S) \xrightarrow{\sim} \partial \otimes S$ as $\mathcal{O}_F \otimes S$-mod.
$*$ $i : \partial \otimes \mu_N \hookrightarrow X$ a $\Gamma_{00}(N)$-level structure, s.t.

(MF1) $\mathcal{F}$ depends on the S-iso. class of the 4-tuple, only
(MF2) $\mathcal{F}$ commutes with base change $S \to S'$ (i.e. $\mathrm{Spec}\, S' \to \mathrm{Spec}\, S$)
(MF3) $\forall\, \alpha \in S^\times$, $\mathcal{F}(X/_S, \lambda, \alpha\omega, i) = \alpha^{-k} \mathcal{F}(X/_S, \lambda, \omega, i)$.

2) Suppose R (= the base) is a $\mathbb{Z}_p$-algebra. Then a <u>p-adic</u> modular form $\mathcal{E}$ of level $\Gamma_{00}(N)$ is a rule $(X/_S, \lambda, \underline{i}) \xrightarrow{\mathcal{E}} S$ as above, but

$*$ with NO WEIGHT $\Leftrightarrow$ NO ω
$*$ $\underline{i}$ is a compatible system of $\Gamma_{00}(Np^h)$-structures, $h \geq 1$.

s.t. (MF1) & (MF2) hold.

$\rightsquigarrow$ Denote by $M(\Gamma_{00}(N), k; R)$ and $V(\Gamma_{00}(N); R)$ the R-modules of modular (resp. p-adic modular forms).

Rmk · Observe that, given $\underline{i} = (i_h)_{h\geq 1}$ we deduce an iso (by taking the inductive limit) $\partial \otimes \widehat{\mathbb{G}}_m \xrightarrow{\sim} \widehat{X} \Rightarrow \mathrm{Lie}(-)$

gives $\omega_{\underline{i}} : \partial \otimes R \xrightarrow{\sim} \mathrm{Lie}(X/R)$. We deduce a map
$M(\Gamma_{00}(N), k; R) \xrightarrow{\vartheta} V(\Gamma_{00}(N); R)$, $(\vartheta f)(X/S, \lambda, \underline{i}) = f(X/S, \lambda, \omega_{\underline{i}}, i_0)$

- Great advantage of p-adic forms: they have no weight!
  $\Rightarrow$ If f has weight $k \neq 0$, and $a \in R$, $(f - a)$ is not a classical m.f. but it is still a p-adic one! Remember...
- Deligne-Ribet and Katz work with c-polarizations $\leadsto$ slightly different. In applications to p-adic L fct, $c = \mathfrak{d}_F$.
- What is the "meaning"? The connection between M and V?

§ Moduli interpretation & Ribet's theorem

Consider the functor

$$\begin{array}{ccc} \underline{\mathrm{Sch}} & \longrightarrow & \underline{\mathrm{Sets}} \\ \left(\begin{array}{c} S \\ \downarrow \\ \mathrm{Spec}\,\mathbb{Z} \end{array}\right) & \longmapsto & \left(\begin{array}{l} \text{Iso. classes of triples} \\ (X/S, \lambda, i) \text{ w. } X/S \text{ HBAV, } \lambda : X \xrightarrow{\sim} X^t \\ i = \Gamma_{00}(N)\text{-level structure.} \end{array}\right) \end{array}$$

It is represented by an algebraic stack, representable if $N \geq 4$:
$\underline{\text{I cheat}}$ and argue it is representable by a $\underset{\sim}{\text{scheme}}$ $\mathcal{M}(\Gamma_{00}(N))/\mathbb{Z}$

* $\forall R$, restriction to $\underline{\mathrm{Sch}}_R$ is rep. by $\underbrace{\mathcal{M}(\Gamma_{00}(N), R) = \mathcal{M}(\Gamma_{00}(N)) \times_{\mathbb{Z}} R}_{\mathcal{M}}$
* By def, $\forall T \to R$ we have
$$\mathcal{M}(T) \overset{1:1}{\longleftrightarrow} \{\text{Iso. classes of } (X/T, \lambda, i)\}$$
* $\exists\ (\mathcal{X}_{univ}/\mathcal{M}, \lambda_{univ}, i_{univ})$, a $\mathcal{M}$-HBAV + polariz + $\Gamma_{00}(N)$-str. s.t.
$\forall T \to R$, $\forall (X/T, \lambda, i)$, $\exists !\, f : T \to \mathcal{M}$ (i.e. $f \in \mathcal{M}(T)$) s.t.
$(X/T, \lambda, i) = f^*(\mathcal{X}/\mathcal{M}, \lambda_{univ}, i_{univ})$. It corresponds to $id \in \mathcal{M}(\mathcal{M})$.

Theorem (Rapoport) 1) Let $\underline{\omega}(k)$ be $\underline{\omega}_{\mathcal{X}/\mathcal{M}}$ with $(\mathcal{O}_F \otimes \mathcal{O}_{\mathcal{M}})^\times$ action twisted by $\alpha \cdot \eta = \alpha^k \eta$. Then
$M(\Gamma_{00}(N), k; R) = H^0(\mathcal{M}, \underline{\omega}(k))$

2) Suppose $R = \mathbb{Z}_p$-alg, denote by $\widehat{\mathcal{M}}$ = p-adic formal completion then $V(\Gamma_{00}(N); R) = H^0(\widehat{\mathcal{M}}, \widehat{\mathcal{O}}_{\mathcal{M}})$.

3) [Ribet] The geometric fibers of $\mathcal{M} \to \mathrm{Spec}(\mathbb{Z})$ are all geometrically irreducible.

§ q-expansions and Main theorem

Recall that we want to produce a power series
$$F(\mathfrak{a}) = \mathcal{N}(\mathfrak{a})\psi(\mathfrak{a})\left\{ L(0, \varepsilon_{\mathfrak{a}}\psi) + \sum_{\mu \in ?} \varepsilon_{\mathfrak{a}}(\mu) q^{\mu} \right\} \in \mathbb{Q}_p[[q]]$$

$\Rightarrow$ Construct a HBAV over $\mathbb{Q}_p[[q]]$ & $f \in M(\Gamma_{00}(N), k; \mathbb{Z}_p)$ & evaluate!

$\ast$ Tate's HBAV.

- $\mathfrak{a} \subseteq \mathcal{O}_F$ ideal, unramified in M_{∞}/F
- $\mathbb{Z}[[q(\mathfrak{a})]] := \left\{ \sum_{\mu} n_\mu \cdot q^{\mu} : \begin{cases} \mu \in \mathfrak{a}, \ \mu \gg 0, \text{ tot. positive} \\ \exists \ \mathbb{Z}\text{-basis of } \mathcal{O}_F \text{ w.r.t which all coord. } \mu_i > 0 \end{cases} \right\}$

f.g. monoid, unlike $[\mu \in \mathfrak{a}, \mu \gg 0] \overset{\text{''}}{=} \mu \in \mathfrak{a}^+$

$\Downarrow$

$\mathbb{Z}[[q(\mathfrak{a})]]_R = \mathbb{Z}[[q(\mathfrak{a})]] \otimes R$ & $\mathbb{Z}((q(\mathfrak{a})))_R = \mathbb{Z}[[q(\mathfrak{a})]]_R\left[\frac{1}{q^a}\right]$ (any a).

$\Rightarrow$ There is a natural map $\mathfrak{a} \xrightarrow{q} \mathbb{G}_m/_{\mathbb{Z}[[q(\mathfrak{a})]]}$ as $\mathcal{O}_F$-gp schemes by $a \mapsto a$: $\mathbb{G}_m/q(\mathfrak{a})$ admits a structure of rigid analytic space which is induced by a $\mathbb{Z}((q(\mathfrak{a})))$-scheme $\mathrm{Tate}(\mathfrak{a})$. Almost by construction, $\mu_N \hookrightarrow \mathbb{G}_m$ induces a $\Gamma_{00}(N)$ level str. i_{Tate} ($\&$ $\underline{i}_{Tate} \Rightarrow \omega_{Tate}$) $\Rightarrow \left(\mathrm{Tate}(\mathfrak{a})/_{\mathbb{Z}((q(\mathfrak{a})))_R}, \lambda_{Tate}, \omega_{Tate}, i_{Tate}\right)$

$\Rightarrow$ We get the q-expansion of $f \in M(\Gamma_{00}(N), k; R)$ (resp. $\mathcal{E} \in V(\Gamma_{00}(N); R)$) at the cusp determined by $\mathfrak{a}$: "$ev_{\mathfrak{a}}$".

- $\mathfrak{a}$ unramified $\Rightarrow f(\mathrm{Tate}(\mathfrak{a}), \lambda_{Tate}, \omega_{Tate}, i_{Tate}) = (\vartheta f)(\widehat{\mathrm{Tate}(\mathfrak{a})}, \lambda_{Tate}, \underline{i}_{Tate})$
  $\rightsquigarrow V(\Gamma_{00}(N); R)$, although being $H^0(\widehat{\mathcal{M}}, \widehat{\mathcal{O}}_{\mathcal{M}})$ contains all q-exp's of forms in $\hat{H}^0(\mathcal{M}, \underline{\omega}(k))$, $\forall\, k \in \mathbb{Z}$.

Corollary to Ribet's thm (q-expansion principle)

(i) $f \in M(\Gamma_{00}(N), k; R)$, $\mathfrak{a} \subset \mathcal{O}_F$: $e_v(\mathfrak{a})(f) = 0 \Rightarrow f = 0$.

(ii) $[E:\mathbb{Q}_p] < \infty$, $\mathcal{E} \in V(\Gamma_{00}(N), E)$, $\mathfrak{a} \subset \mathcal{O}_F$.

$$ev_{\mathfrak{a}}(\mathcal{E}) \in \mathcal{O}_E((q(\mathfrak{a}))) \Rightarrow \mathcal{E} \in V(\Gamma_{00}(N), \mathcal{O}_E).$$

Sketch: Each $\left(\mathrm{Tate}_{\mathfrak{a}}/_{\mathbb{Z}((q(\mathfrak{a})))_R}, \lambda_{Tate}, \omega_{Tate}, i_{Tate}\right)$ corresponds to a map $\varphi_{\mathfrak{a}}: \mathrm{Spec}(\mathbb{Z}((q(\mathfrak{a})))) \longrightarrow \mathcal{M}$: it can be seen that each image contains the generic point of every fiber. By Rapoport 1), $ev_{\mathfrak{a}}(f) = 0 \Rightarrow f = 0$ on these generic points $\overset{\text{Ribet}}{\Longrightarrow} f = 0$.

(ii) Follows from (i) by studying the exact sequence

$$0 \to V(\Gamma_{00}(N); \mathcal{O}_E) \to V(\Gamma_{00}(N); E) \to E((q(\mathfrak{a})))/\mathcal{O}_E((q(\mathfrak{a})))$$

$\square$

Fix $\varepsilon \in LC(\Pi, \mathbb{Z}_p)$, $j \le 0$: interpreting $M(\Gamma_{00}(N), k; \mathbb{C})$ as functions on lattices, we can produce a Eisenstein series $G_{\varepsilon,\psi,1-j}$ s.t.

$$ev_{\mathfrak{a}}(G_{\varepsilon,\psi,1-j}) = N\mathfrak{a}^{1-j}\psi(\mathfrak{a})\Big\{L(j, \varepsilon_{\mathfrak{a}}\psi) + \sum_{\mu \in (\mathfrak{a}^2)^+} \Big[\underbrace{\sum_{I \subseteq \mathfrak{a}^2}}_{\text{FINITE SUM!}} \varepsilon_{\mathfrak{a}}(\mu I^{-1}) N(\mu I^{-1})^{-j}\Big] q^{\mu}\Big\} \in \mathbb{Q}_p(\psi)((q(\mathfrak{a}^2)))$$

Rmk Strictly speaking, the above is only true for $j \leq 1$, and for $j=0$ we need a modification. Observe that for [A], any j (instead of $j=0$) would do, thanks to [B]. Therefore, we will pretend that also $G_{\varepsilon,\psi,0}$ exists and satisfies the above equation at ev_2: we write $G_{k,\varepsilon} := G_{k,\varepsilon,0}$.

$\Downarrow$

Look at $G_{\varepsilon,\psi} \in V(T_{\infty}(N); \mathbb{Q}_p(\psi))$ and $G'_{\varepsilon,\psi} = G_{\varepsilon,\psi} - (Nc)\psi(c) L(0, \varepsilon_c \psi)$.

As p-adic modular forms have no weight, $G'_{\varepsilon,\psi} \in V(T_{\infty}(N), \mathbb{Q}_p(\psi))$.

By Cor (ii), $G'_{\varepsilon,\psi} \in V_{\infty}(T(N), \mathbb{Z}_p(\psi))$,

$$\Rightarrow ev_1(G'_{\varepsilon,\psi}) = ev_1(G_{\varepsilon,\psi}) - (Nc)\psi(c) L(0, \varepsilon_c \psi) =$$

$$L(0, \varepsilon\psi) - (Nc)\psi(c) L(0, \varepsilon_c \psi) + \sum_{\mu} [\dots] q^{\mu} \in \mathbb{Z}_p[\psi]((q(\mathfrak{d}^{-1})))$$

$\Rightarrow$ [A].