p-adic L functions for totally real number fields III

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§ Introduction

Recall: F tot. real, $n := [F:\mathbb{Q}]$, M/F abelian, $\Delta = \mathrm{Gal}(M/F)$

<u>M tot. real</u>, $M \cap F_\infty = F$, $G = \mathrm{Gal}(M_\infty/F)$, $\psi : \Delta \to \overline{\mathbb{Q}}^\times \left(\begin{smallmatrix} \hookrightarrow \mathbb{C}_p \\ \hookrightarrow \mathbb{C} \end{smallmatrix} \right)$.

$\forall\, \varepsilon \in LC(\Gamma, \mathbb{C})$, $L(\varepsilon\psi, s)$ is the meromorphic continuation of

$$s \mapsto \sum_{\substack{0 \neq \mathfrak{a} \subseteq \mathcal{O}_F \\ \text{unr. in } M_\infty/F}} \frac{\varepsilon(\mathfrak{a})\psi(\mathfrak{a})}{N\mathfrak{a}^s}, \quad \mathrm{Re}(s) > s_0 \quad (s_0 = s_0(\varepsilon)).$$

ε loc. const $\Rightarrow \exists\, H \trianglelefteq_{\circ} G$: $\varepsilon\psi : G/H \to \mathbb{C}$. Then

$$L(\varepsilon\psi, s) = \sum_{g \in G/H} \varepsilon(g)\psi(g) \cdot L(\chi_{\tilde{g}+H}, s),$$ $\chi_\bullet$ = char. function, $\tilde{g}$ = lift of g.

Shrink $H \Rightarrow$ can assume $H = \Gamma^{p^n}$, $\Gamma = \mathrm{Gal}(F_\infty/F) \Rightarrow \Gamma^{p^n} = \mathrm{Gal}(F_\infty/F_n)$.

F is "arbitrary" $\rightsquigarrow$ replace $F \leftrightarrow F_n$, $n \leftrightarrow 0$. So, enough to consider

$$L(\chi_{\tilde{g}+\Gamma}, s) = \sum_{\substack{0 \neq \mathfrak{a} \subseteq \mathcal{O}_F \\ \text{unr.} \\ (\mathfrak{a}, M/F) = \mathrm{res}_M(\tilde{g})}} \frac{1}{N\mathfrak{a}^{-s}} := \zeta(\sigma, s) \quad \text{if } \sigma = \mathrm{res}_M(\tilde{g}) \in \Delta.$$

↳ Hurwitz partial zeta.

$[\mathcal{A}']$ $\zeta(\sigma, 0) - N(\mathfrak{c})\,\zeta(\sigma \cdot (\mathfrak{c}, M/F), 0) \in \mathbb{Z}_p$ $\forall\, \sigma \in \Delta$ ($\mathfrak{c}$ unr. in M_∞/F)

Congruence $[\mathcal{B}]$ involves η and ε crucially $\rightsquigarrow$ in terms of $\zeta(\sigma, s)$?

<u>Remember</u>: $[\mathcal{B}]$ only used to compute $\int k_{cyc}^{1-j}\, \varepsilon\, d\lambda_\psi \overset{?}{\doteq} L(j, \varepsilon\psi)$.

Cassou-Noguès & Barsky } Another strategy.

Remember $\Gamma_p : \mathcal{M}(\Gamma, \mathbb{Q}_p(\psi))^{tor} \longrightarrow \mathcal{C}(\mathbb{Z}_p, \mathbb{C}_p)$. Let $\mathcal{J} = \mathrm{Im}(\Gamma_p)$. Then

consider $\xi_\sigma : j \longmapsto \zeta(\sigma \cdot (\mathfrak{c}, M_\infty/F)^{-1}, j) - (N\mathfrak{c})^{1-j} \cdot \zeta(\sigma, j)$, for $j \leq 0$, $\sigma \in \Delta$

[C] ξ_σ is continuous; it can be extended to $\xi_\sigma(s) \in \mathcal{G} \subseteq \mathcal{C}(\mathbb{Z}_p, \mathbb{C}_p)$

Prop 1: Assume [A']. Then [C] implies that

$$\int K_{cyc}^{1-j} \cdot \phi \, d\zeta_{p,c} = \left(1-(Nc)^{1-j} \cdot \phi(c)\right) L(\phi, j) \quad \forall j \leq 0, \forall \underset{\text{finite order}}{\phi : \Gamma \to \overline{\mathbb{Q}}_p^\times}$$

Pf. Let $\mu := T_p^{-1}(\xi(s))$. Then $\forall \phi \in \mathrm{Hom}_{cont}(\Gamma, \overline{\mathbb{Q}}_p^\times)$ finite order,

$$\int \phi \, d\mu = \cancel{\phi(\gamma)}\cancel{\phi(\gamma)^{-1}} \zeta(\sigma, 0) - N(c)\phi(\gamma)\zeta(\sigma, \Theta) = \int \phi \, d\zeta_{p,c}$$

$\Rightarrow \mu = \zeta_{p,c}$. But, by definition of T_p, then $\int K_{cyc}^{1-j} \phi \, d\zeta_{p,c} =$

$\xi_{\phi \cdot \sigma}(j) = \left(1-(Nc)^{1-j} \cdot \phi(c)\right) L(\phi, j) \quad \forall \phi, j \leq 0.$ $\square$

Here σ stands for $\chi_{\widetilde{\sigma+\Gamma}} \Rightarrow \phi \cdot \sigma \leftrightarrow \phi \cdot \chi_{\widetilde{\sigma+\Gamma}}$.

$\Rightarrow$ Need to study in detail $\zeta(\sigma, j)$ for $j \in \mathbb{Z}_{\leq 0}$ (the explicit definition is for $\mathrm{Re}(s) > 1$...)

§ Holomorphic continuation

Main result (proven by Barsky, Cassou-Noguès, Colmez, Shintani...)

Thm. Let $\underline{v} = v_1, \ldots, v_r \in (\mathbb{R}_+)^n$, $L = L_{\underline{v}} : \mathbb{R}^r \to \mathbb{R}^n$ the linear function $(y_1, \ldots, y_r) \mapsto \sum_{i=1}^r y_i \cdot v_i$. Let $\underline{\xi} = (\xi_1, \ldots, \xi_r)$ roots of 1,

$$Z(L, \underline{\xi}; s) = \sum_{\underline{m} \in \mathbb{N}^r} \frac{\xi_1^{m_1} \cdots \xi_r^{m_r}}{\mathrm{Norm}(L(\underline{m}))^s}, \quad \mathrm{Re}(s) > 1. \quad \left(\mathrm{Norm}(\underline{x}) = \prod x_i\right)$$

Then $Z(L, \underline{\xi}; s)$ can be meromorphically extended to $\mathbb{C}$, even holomorphically if $\xi_i \neq 1 \; \forall i$; and

$$Z(L, \underline{\xi}; j) = R_{L,j}(\underline{\xi}) \qquad \forall j \leq 0$$

where $R_{L,j}(T) \in \mathbb{R}(T)$ is $R_{L,j}(T) = \sum_{\underline{m} \in \mathbb{N}^r} \mathrm{Norm}(L^{-j}(\underline{m})) \cdot T^{\underline{m}}$.

<u>Rmk</u> This says that the "wrong" substitution for $j \le 0$ in the expression defining $Z(L, \xi; s)$ for $\mathrm{Re}(s) > 1$ is "right" when replacing a divergent power series by the corresponding rational function (whose existence is <u>not</u> obvious).

$\Rightarrow$ No divergency! But may be poles...

<u>Idea of Pf</u>: The main tool is the <u>following</u> <u>classical</u> $(\mathrm{char}(A) \neq 0)$

<u>Lemma</u> If $P(X) \in A[X_1, \ldots, X_r]$, then $\sum_{n \in \mathbb{N}^r} P(n) T^n \in A[[T_1, \ldots, T_r]]$ coincides with $\sum_{\substack{k \in \mathbb{N}^r \\ k \neq (0, \ldots, 0)}} \frac{\lambda_k}{(1-T_1)^{k_1} \cdots (1-T_r)^{k_r}}$ with $\lambda_k = \begin{pmatrix} \text{binomial} \\ \text{coeff.} \end{pmatrix} \cdot P(k)$

<u>almost all 0</u>

$\Rightarrow$ Apply this to a certain polynomial constructed by Fourier transform of L: in particular, if $v_1, \ldots, v_r \in (\mathbb{Z}_{>0})^n \Rightarrow$ this has coeff. in $\mathbb{Z} \Rightarrow \lambda_k \in \mathbb{Z} \ \forall\, k \in \mathbb{N}^r$.

$\square$

<u>Cor</u> With notations as in the Thm, suppose $v_1, \ldots, v_r \in (\mathbb{Z}_{>0})^n$ and $\xi_1, \ldots, \xi_r$ are of order $c > 1$, $(p, c) = 1$. Then $Z(L, \xi; j) \in \mathbb{Z}_p$ $\forall\, j \le 0$

<u>Pf</u> $Z(L, \underline{\xi}; j) = R_{L,j}(\underline{\xi}) = \sum_{\substack{k \in \mathbb{N}^r \\ \text{s.t.} \\ \lambda_k \neq 0}} \frac{\lambda_k}{\prod_i (1-\xi_i)^{k_i}}$. It is a finite sum with $\lambda_k \in \mathbb{Z}$ and $v_p(1-\xi_i) = 0$ because $(p, c) = 1$.

$\square$

$\Rightarrow$ Need to relate $\zeta(\sigma, j)$ to $Z(L, \underline{\xi}; j)$ for suitable L i.e. we need to transform the sum over all <u>ideals</u> in $\mathcal{O}_F$ unramified in M_∞/F into a sum over $\underline{m} \in \mathbb{Z}^n$ of a linear form $L(\underline{m})$.

Rmk We actually relate the "twisted zeta-value" $\zeta(\sigma,j)-N(c)\zeta(\sigma\cdot(c,M/F),j)$ to $Z(L,\underline{\xi};j)$ with $\mathrm{ord}(\xi_i)=c$, where $c=Nc \Rightarrow$ Need $c\neq 1$ and $(c,p)=1$.

§ Shintani method

- Want to produce linear forms and relate their Z to ζ!

Recall $\sigma\in\Delta$, $\zeta(\sigma,s)=\sum_{\substack{(fp,\mathfrak{a})=1\\(\mathfrak{a},M/F)=\sigma}}\frac{1}{N\mathfrak{a}^{-s}}$, if $\mathrm{Re}(s)>1$

where $f=\mathrm{cond}(M/F)$.

$Cl(fp)^+=\{\text{fract. ideals prime to } fp\}\Big/\left\{\mathfrak{a}=(\alpha) \text{ with } \begin{array}{l}\alpha\equiv 1\ (fp)\\ \alpha\gg 0\end{array}\right\}\leftarrow$ finite!

$\forall\ R\in Cl(fp)^+$ fix a rep. $\mathfrak{b}_R\subseteq\mathcal{O}_F$.

$$\Rightarrow \zeta(\sigma,s)=\underbrace{\sum_{\substack{R\in Cl(fp)^+\\(R,M/F)=\sigma}}}\ \sum_{\mathfrak{a}\sim\mathfrak{b}_R}\frac{1}{N\mathfrak{a}^{-s}}=\sum_{\substack{R\in Cl(fp)^+\\(R,M/F)=\sigma}}(N\mathfrak{b}_R)^s\sum_{\substack{\alpha\equiv 1\ (fp)\\ \alpha\gg 0\\ \alpha\in\mathfrak{b}^{-1}}\Big/E(fp)^+}\frac{1}{|N\alpha|^s}.$$

Only depends on $Cl(fp)^+$

$\mathfrak{a}\sim\mathfrak{b}_R \Leftrightarrow \mathfrak{a}\mathfrak{b}_R^{-1}=(\alpha)$

$E(fp)^+=\left\{\varepsilon\in\mathcal{O}_F^\times,\ \varepsilon\gg 0,\ \varepsilon\equiv 1\ (fp)\right\}$

Well-def, $|N\varepsilon|=1$ $\forall\ \varepsilon\in E(fp)^+$.

Since we can alway chose $(N\mathfrak{b}_R,p)=1$, we get

Prop. 2 $\zeta(\sigma,j)\in\mathbb{Z}_p \Longleftrightarrow$ analytic continuation of

$$s\longmapsto\sum_{\substack{\alpha\gg 0\\ \alpha\equiv 1\ (fp)\\ \alpha\in\mathfrak{b}^{-1}}\Big/E(fp)^+}\frac{1}{|N\alpha|^s} \text{ is } p\text{-integral at } j.$$

Def A cone in $F\otimes\mathbb{R}$ (or in F, by abuse of notation) is a set $C(v_1,\dots,v_r)=\left\{\sum_{i=1}^r\lambda_i\cdot v_i : \lambda_i>0\right\}\subseteq(F\otimes\mathbb{R})^+$ for $v_i\in(F\otimes\mathbb{R})^+$.

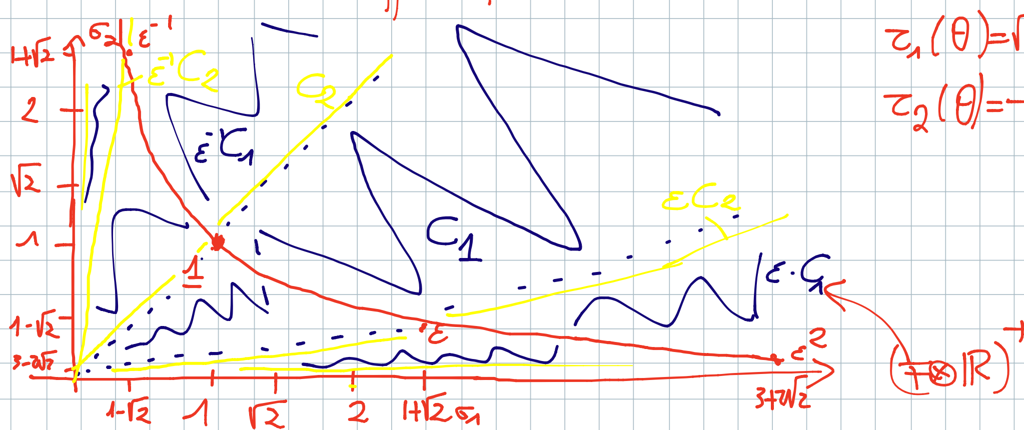
Thm (Shintani) There exist finitely many disjoint cones C_i in F s.t. $(F\otimes\mathbb{R})^+/E(fp)^+ \overset{1:1}{\longleftrightarrow} \bigsqcup_{i=1}^{t} C_i$; in other words, every

$\alpha \gg 0$ is $\alpha = \eta \cdot v$ with $\eta \overset{!}{\in} E(fp)^+$, $v \overset{!}{\in} C_i$.

Ex $F = \mathbb{Q}(\theta)$, $\theta^2 = 2$. Suppose $fp = 1 \Rightarrow Cl^+ = \{1\}$. Then $\tau_i : \hookrightarrow \mathbb{R}$

$\tau_1(\theta) = \sqrt{2}$

$\tau_2(\theta) = -\sqrt{2}$



Application: $\displaystyle\sum_{\substack{\alpha \gg 0 \\ \alpha \equiv 1 (fp) \\ \alpha \in b^{-1}}} \frac{1}{|N\alpha|^s} = \sum_{i=1}^{t} \sum_{v \in C_i \cap 1 + b^{-1}fp} \frac{1}{|\mathrm{Norm}(v)|^s}$

Similarly, $s \mapsto \zeta(\sigma, s) - (Nc)^{1-s}\zeta(\sigma\cdot(c, M/F), s)$ can be expressed

as $\displaystyle\sum_{i=1}^{t} \sum_{v \in C_i \cap 1 + bp^{-1}f} \frac{\exp(\mathrm{Tr}(v \cdot \xi))}{N(v)^s}$ for suitable roots of 1 ξ of order $c = N_{\mathbb{Q}}^{F}(c)$.

Then the "twisted" analogue of Prop. 2 is

Prop 3. [A'] is true $\Longleftrightarrow$ the analytic continuation of $s \mapsto \displaystyle\sum_{v \in C_i \cap 1 + bp^{-1}f} \frac{\exp(\mathrm{Tr}(v \cdot \xi))}{N(v)^s}$ evaluated at $s = 0$ is integral at p, for each $1 \leq i \leq t$.

Therefore, we want to use Cor, which gave conditions for integrality: let $C_i = \left\{ \sum_{i=1}^{r(\cdot)} w_i \cdot \lambda_i : \lambda_i > 0 \right\}$. As $C_i \cap 1 + b^{-1} f p$ is a lattice $\Rightarrow \exists$ basis $\{v_i\}$ of C_i s.t. $C_i \cap 1 + b^{-1} p f = \left\{ \sum_{i=1}^{r} v_i \cdot n_i,\ n_i \in \mathbb{Z}_{>0} \right\}$

and $v_i \in \mathcal{O}_F \Rightarrow \displaystyle\sum_{v \in C_i \cap 1 + b^{-1} p f} \frac{\exp(\mathrm{Tr}(v \cdot \xi))}{(\mathrm{Norm}(v))^s} = Z(L(v_1, \ldots, v_r), \xi; s)$

$\in \mathbb{Z}_p$ by Corollary!

$\square$

As for [C], it is almost immediate to check that $\mathcal{O} = \overline{E(\mathbb{Z}_p, \mathbb{C}_p)}$ in the sup norm, where E are the "exponential functions" $E(\mathbb{Z}_p, \mathbb{C}_p) = \{$finite sums of $s \mapsto u^s\}$

Since the coefficients λ_k in Lemma are binomial coeff.'s, $s \mapsto \lambda_s$ is in $\mathcal{O}$; and $v_p(\xi - 1) = 0$ shows $s \mapsto R_{L,s}(T)$ has the right analyticity condition.