

p -adic L -functions for CM fields^{*1}

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INTRODUCTION OF p -ADIC L -FUNCTIONS FOR CM-FIELDS

PRELIMINARIES

PERIODS

MAIN THEOREM.

(STRATEGY $\rightarrow$ DAY 2)

PRELIMINARIES

DEFINITION

A number field K is called a CM-FIELD if K is a totally imaginary quadratic extension of a totally real number field F .

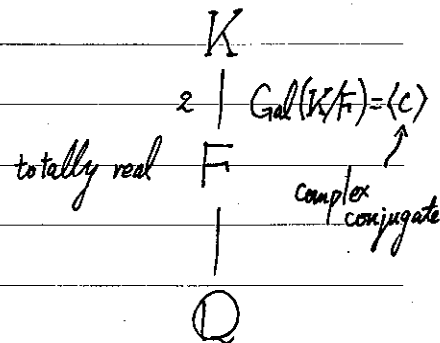
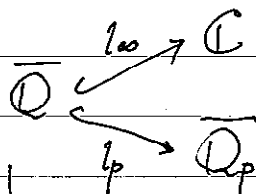
p : prime number

Fix embeddings

$I_K := \{ \sigma: K \hookrightarrow \mathbb{C} \}$
embeddings

$\cup$

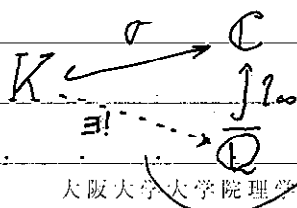
Σ subset



DEFINITION

$(K, \Sigma): \text{CM-TYPE} \stackrel{\text{def}}{\iff} (K: \text{CM-field and})$
 $I_K = \sum \cup \Sigma^c, \quad \sum \cap \Sigma^c = \emptyset$
 $(\Sigma^c := \{ \sigma \circ c \mid \sigma \in \Sigma \})$

$\Rightarrow$ REMARK.



We also denote σ (by abuse of notation)

$$\Sigma_p := \{ \mathfrak{P} \subseteq \mathcal{O}_K \mid \text{prime ideal above } p \text{ induced by } K \xrightarrow{\sigma} \overline{\mathbb{Q}} \xrightarrow{\tau} \overline{\mathbb{Q}}_{\mathfrak{P}} \forall \sigma \in \Sigma \}$$

$$\Sigma_p^c = \overline{\Sigma}_p := \{ \mathfrak{P} \subseteq \mathcal{O}_K \mid \text{prime ideal above } p \text{ induced by } K \xrightarrow{\sigma \circ c} \overline{\mathbb{Q}} \xrightarrow{\tau} \overline{\mathbb{Q}}_{\mathfrak{P}} \forall \sigma \in \Sigma \}$$

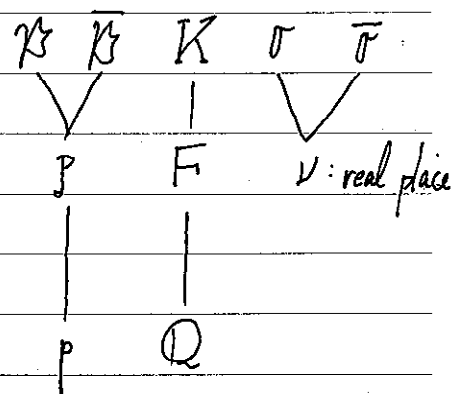
DEFINITION

A CM-type (K, Σ) is p-ORDINARY

$$\stackrel{\text{def}}{\Leftrightarrow} \{ \text{prime ideals of } \mathcal{O}_K \text{ above } p \} = \Sigma_p \sqcup \overline{\Sigma}_p, \Sigma_p \cap \overline{\Sigma}_p = \emptyset$$

REMARK K : CM-type.

$$\exists \text{ p-ordinary CM-type} \Leftrightarrow \forall \mathfrak{P} \subseteq \mathcal{O}_F \text{ above } p \text{ splits in } K$$



If exists,

$$\# \{ \text{p-ordinary CM-types} \} = 2^{\# \{ \mathfrak{P} \subseteq \mathcal{O}_F \text{ above } p \}} \leq 2^{[F:\mathbb{Q}]} = \# \{ \text{CM-types} \}$$

"which $\mathfrak{P}$ you choose for each $\mathfrak{p} \subseteq \mathcal{O}_F$ " "which σ you choose for each real place ν of F "

DEFINITION

K : CM field. A character $\lambda: A_K^\times / K^\times \rightarrow \mathbb{C}^\times$ is called

a GRÖSSENCHARACTER OF TYPE (A_0) of infinite type $\kappa = \sum_{\sigma \in I_K} \kappa_\sigma \sigma \in \mathbb{Z}[I_K]$
(or ALGEBRAIC HECKE CHARACTER)

if

$$\lambda_{A_0}(z_{A_0}) = \prod_{\nu} z_\nu^{\kappa_{\sigma_\nu}} \overline{z}_\nu^{\kappa_{\overline{\sigma}_\nu}} \quad \forall z \in A_K^\times$$

ν : archimedean places $(= \sigma_\nu, \overline{\sigma}_\nu)$

⇒ REMARK

(1) $\lambda: A_K^*/K^* \rightarrow \mathbb{C}$ Größencharacter of type (A_0)
of infinite type $\kappa = \sum_{\sigma \in I_K} \kappa_\sigma \sigma$

⇒ Automatically we have $\kappa_{\sigma_v} + \kappa_{\bar{\sigma}_v} \equiv k \in \mathbb{Z}$ (independent of v)

So if we fix a CM-type Σ , we can uniquely rewrite

$$\kappa = \underbrace{k}_{1 \text{ variable}} \Sigma + \sum_{\sigma \in \Sigma} \underbrace{d_\sigma}_{[F:\mathbb{Q}] \text{-variable}} (\sigma - \bar{\sigma}) \quad k, d_\sigma \in \mathbb{Z} \quad (\forall \sigma \in \Sigma).$$

($\# \Sigma = [F:\mathbb{Q}]$) ($k\Sigma := k \sum_{\sigma \in \Sigma} \sigma$)

in this case, λ is called "of weight $-k$ " (as a pure motif)

(2) $\lambda: A_K^*/K^* \rightarrow \mathbb{C}$ of type (A_0) of infinite type $\kappa = k\Sigma + \sum_{\sigma \in \Sigma} d_\sigma (\sigma - \bar{\sigma})$

⇒ $\text{Im}(\lambda|_{A_{K, \text{fin}}^*}) \subseteq \exists L/\mathbb{Q}$ finite extension.

so we can define:

DEFINITION λ : as above of infinite type $\kappa = k\Sigma + \sum_{\sigma \in \Sigma} d_\sigma (\sigma - \bar{\sigma})$

⇒ define $\hat{\lambda}: A_K^*/K^* \rightarrow \underline{\mathbb{Q}_p}^*$ by.

$$\hat{\lambda}(x) := \underbrace{z_p(\lambda(x) x_\infty^{-\kappa})}_{\in \underline{\mathbb{Q}} \text{ by the above remark}} \times \prod_{\mathfrak{p} \in \Sigma_f} \prod_{\substack{\sigma: K \hookrightarrow \bar{\mathbb{Q}} \\ \sigma_p := \sigma \\ \text{induces } \mathfrak{p}}} \sigma_p(x_{\mathfrak{p}})^{\kappa_{\mathfrak{p}}} \bar{\sigma}_p(x_{\bar{\mathfrak{p}}})^{\kappa_{\bar{\mathfrak{p}}}}$$

$\hat{\lambda}$ is called the p-ADIC AVATAR OF λ

PERIODS

 (K, Σ) p -ordinary CM-type

$$\nu \in I_F := \{\nu: F \hookrightarrow \mathbb{R} \text{ embeddings}\}$$

$$\downarrow \quad \downarrow$$

$$\sigma \in \Sigma$$

$$\text{s.t., } \sigma|_F = \nu$$

via this, identify

$$F \otimes \mathbb{C} \simeq \mathbb{C}^{I_F}$$

$$\downarrow$$

$$K \otimes \mathbb{R} \simeq \mathbb{C}^\Sigma$$

GOAL $\pi \subseteq F$ fractional ideal prime to p construct π -polarized HILBERT-BLOUMENTHAL abelian variety
of level $\Gamma_0(p^\infty)$

$$(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K)) / \mathbb{C}$$

WITH COMPLEX MULTIPLICATION BY $\mathcal{O}_K$

$$\bullet X(\mathcal{O}_K) = \mathbb{C}^{I_F} / \mathcal{O}_K \quad \left[\begin{array}{l} \text{via above identification,} \\ \mathcal{O}_K \text{ can be regarded as a lattice of } \mathbb{C}^{I_F} \end{array} \right]$$

: complex torus

$$\bullet \pi\text{-polarization : i.e., } \Lambda_{\mathcal{O}_F}^2(\mathcal{O}_K) \xrightarrow{\sim} \pi^{-1} \mathcal{O}_F^{-1} \quad (\mathcal{O}_F: \text{different of } F)$$

(*) Choose $\delta \in K$ purely imaginary (i.e., $\bar{\delta} = -\delta$) s.t.,

$$\left\{ \begin{array}{l} \bullet \text{ the image of } \delta \otimes 1 \in K \otimes \mathbb{R} \xrightarrow{\sim} F \otimes \mathbb{C} \text{ is contained} \\ \text{in } F \otimes \mathbb{R} \text{ and totally positive (in } F \otimes \mathbb{R}) \\ \bullet \text{ alternating pairing on } \mathcal{O}_K: \langle u, v \rangle_\delta := \frac{\bar{u}v - u\bar{v}}{2\delta} \text{ defines an isomorphism} \\ \text{of invertible } \mathcal{O}_F\text{-modules: } \Lambda_{\mathcal{O}_F}^2(\mathcal{O}_K) \xrightarrow{\langle \cdot, \cdot \rangle_\delta} \pi^{-1} \mathcal{O}_F^{-1} \end{array} \right.$$

$\rightsquigarrow (X(\mathcal{O}_K), \langle \cdot, \cdot \rangle_\delta)$ (κ -polarized complex torus) defines

a κ -polarized HBAV $(X(\mathcal{O}_K), \lambda(\delta)) / \mathbb{C}$

$$\lambda(\delta) : X(\mathcal{O}_K) \longrightarrow X(\mathcal{O}_K)^t_{\mathcal{O}_F} \otimes \kappa$$

• $\Gamma_\infty(p^\infty)$ -level structure : i.e., $\delta_F^{-1} \otimes \varprojlim_n \mu_{p^n} \hookrightarrow \mathcal{O}_K \otimes \mathbb{Z}_p$

$$\text{Set } \varphi_\Sigma : \mathcal{O}_K \otimes \mathbb{Z}_p \longrightarrow \prod_{\beta \in \Sigma} \hat{\mathcal{O}}_{K, \beta} \xrightarrow{\sim} \prod_{\substack{p \in \mathcal{O}_F \\ \text{above } p}} \hat{\mathcal{O}}_{F, p} = \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

$$\circ \quad \mathcal{O}_K \otimes \mathbb{Z}_p \xrightarrow{\sim} (\mathcal{O}_F \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p)$$

$$\rightsquigarrow \begin{matrix} \psi \\ \downarrow \\ \alpha \end{matrix} \longmapsto (\varphi_\Sigma(x), \varphi_\Sigma(\bar{x}))$$

$$\circ \quad \mathcal{O}_F \otimes \mathbb{Z}_p \xrightarrow{(*)} \delta_F^{-1} \otimes \mathbb{Z}_p$$

$\therefore$

$$\Lambda^2_{\mathcal{O}_F \otimes \mathbb{Z}_p}(\mathcal{O}_K \otimes \mathbb{Z}_p) \xrightarrow{\sim} \mathcal{O}_F \otimes \mathbb{Z}_p$$

Since $\mathcal{O}_K \otimes \mathbb{Z}_p \simeq (\mathcal{O}_F \otimes \mathbb{Z}_p)^2$

$\langle \cdot, \cdot \rangle_\delta$

$$\delta_F^{-1} \kappa^{-1} \otimes \mathbb{Z}_p = \delta_F^{-1} \otimes \mathbb{Z}_p \quad \square$$

κ : prime to p

$$\circ \quad \mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n \mathbb{Z} \xrightarrow{\sim \exp} \varprojlim_n \mu_{p^n} : (a_n)_{n \geq 1} \mapsto \left(\exp(2\pi \sqrt{-1} \cdot \frac{a_n}{p^n}) \right)_{n \geq 1}$$

Combining them,

$$(\mathcal{O}_F \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \xleftarrow{(\varphi_\Sigma(x), \varphi_\Sigma(\bar{x}))} \mathcal{O}_K \otimes \mathbb{Z}_p$$

$$\downarrow \text{(*) for first component}$$

$$(\delta_F^{-1} \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p)$$

$$\uparrow \delta_F^{-1} \otimes \mathbb{Z}_p$$

$$\downarrow \exp$$

$$\delta_F^{-1} \otimes \varprojlim_n \mu_{p^n}$$

$$i(\mathcal{O}_K) : \delta_F^{-1} \otimes \varprojlim_n \mu_{p^n} \hookrightarrow \mathcal{O}_K \otimes \mathbb{Z}_p$$

$\Gamma_\infty(p^\infty)$ -level structure.

So we have constructed $(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K)) / \mathbb{C}$

THEORY OF
COMPLEX
MULTIPLICATION

• $(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K))$ is defined over $\overline{\mathbb{Q}}$

• at the prime of $\mathcal{O}_{\overline{\mathbb{Q}}}$ induced by $\mathcal{O}_{\overline{\mathbb{Q}}} \hookrightarrow \mathbb{Q}_p$,

$(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K))$ has a good ordinary reduction.

i.e., has a model over

$$A = \{x \in \overline{\mathbb{Q}} \mid \iota_p(x) \in \mathbb{D}_p\}.$$

$\mathbb{D}_p$: valuation ring of $\overline{\mathbb{Q}_p} := \widehat{\overline{\mathbb{Q}_p}}$

$$(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K)) / \mathbb{C}$$

$$(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K)) / \mathbb{D}_p$$

$\mathbb{C}^{\Sigma} \xrightarrow{\iota} X(\mathcal{O}_K)$
 $(dz_{\sigma})_{\sigma \in \Sigma} \xleftarrow{\omega_{\text{trans}}}$

$(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K)) / A$

choose $\omega \in \underline{\omega}_{X(\mathcal{O}_K)/A}$

$H^0(X(\mathcal{O}_K), \Omega_{X(\mathcal{O}_K)/A})$ "invariant 1-form"

locally free
 $\mathcal{O}_F \otimes A$ -modules
of rank 1

DEFINITION

• on $\mathbb{C}$, $\omega = \exists \Omega_{\text{CM}} \omega_{\text{trans}}$ $\Omega_{\text{CM}} = (\Omega_{\text{CM}}(\sigma))_{\sigma \in \Sigma} \in (\mathcal{O}_F \otimes \mathbb{C})^{\times} \simeq (\mathbb{C}^{\times})^{\Sigma}$

CM-PERIOD

• on $\mathbb{D}_p$, $\omega = \exists \Omega_p \omega_{\text{can}}$ $\Omega_p = (\Omega_p(\sigma))_{\sigma \in \Sigma} \in (\mathcal{O}_F \otimes \mathbb{D}_p)^{\times} \simeq (\mathbb{D}_p^{\times})^{\Sigma}$

p-ADIC PERIOD

⇒ REMARK.

each Ω_{CM}, Ω_p DOES depend on the choice of ω ,

BUT the "ratio" of (Ω_{CM}, Ω_p) is independent of ω

⇒ SUPPLEMENTARY COMMENTS (definitions of $\omega_{trans}, \omega_{can}$)

By the duality $\underline{\omega}_{X(\mathcal{O}_K)/R} \otimes_{\mathcal{O} \otimes R} \text{Lie}(X(\mathcal{O}_K)/R) \xrightarrow{\sim} \mathcal{O}_F^{-1} \otimes_{\mathbb{Z}} R$,

to take an invariant 1-form $\omega \in \underline{\omega}_{X(\mathcal{O}_K)/R}$ is equivalent to determine

an isomorphism $\text{Lie}(X(\mathcal{O}_K)/R) \xrightarrow{\sim} \mathcal{O}_F^{-1} \otimes_{\mathbb{Z}} R$.

1° ω_{trans} ($R = \mathbb{C}$ case)

By construction, $\text{Lie}(X(\mathcal{O}_K)/\mathbb{C}) = \text{Lie}(X(\mathcal{O}_K)^{an}) = \text{Lie}(\overset{F \otimes \mathbb{C}}{\underset{\text{SI}}{\mathbb{C}^{IF}}}/\mathcal{O}_K)$

$$= F \otimes \mathbb{C} = \mathcal{O}_F^{-1} \otimes \mathbb{C}$$

ω_{trans} is induced by this isomorphism (equality).

2° ω_{can} ($R = \mathbb{D}_p$ case)

The level structure $i(\mathcal{O}_K)$ induces $\mathcal{O}_F^{-1} \otimes \widehat{G}_m \xrightarrow{\sim} \widehat{X}$
as formal schemes / $\mathbb{D}_p$.

This induces on Lie groups

$$\begin{array}{ccc} \mathcal{O}_F^{-1} \otimes \text{Lie}(\widehat{G}_m/\mathbb{D}_p) & \xrightarrow{\sim} & \text{Lie}(\widehat{X}/\mathbb{D}_p) \\ \parallel & & \parallel \\ \mathcal{O}_F^{-1} \otimes \mathbb{D}_p & & \text{Lie}(X/\mathbb{D}_p) \end{array}$$

ω_{can} is induced
by the (inverse of the)
left isomorphism

§ MAIN THEOREM ([K78, THEOREM 5.30] [HT93, THEOREM II, THEOREM 4.1])

(K, Σ) : p -ordinary CM-type.

choose $\delta \in K$ satisfying $(*)$

$\Rightarrow \exists!$ $\mathbb{D}_p$ -valued measure μ on $Cl_K(p^\infty)$
(ray class group modulo p^∞)

s.t.,

$$\frac{\int_{Cl_K(p^\infty)} \hat{\chi} d\mu}{\Omega_p^{k\Sigma+2d}} = [O_K^\times : O_F^\times] w_p(\chi) \frac{(-1)^{[F:\mathbb{Q}]} \pi^{|d|}}{\sqrt{d_F} \operatorname{Im}(\delta)^d} \times \frac{\Gamma_\Sigma(k\Sigma+d)}{\Omega_{CM}^{k+2d}}$$

$$\times \prod_{\beta \in \Sigma_p} (1 - \chi(\beta))(1 - \check{\chi}(\beta)) L(\chi, 0)$$

for all Größencharacter of type (A_0) χ whose conductor dividing p^∞

of type $\kappa = k\Sigma + \sum_{\sigma \in \Sigma} d_\sigma(\sigma - \bar{\sigma})$ satisfying

$$(\#) \left\{ \begin{array}{l} k \geq 1, d_\sigma \geq 0 \quad \forall \sigma \in \Sigma \\ \text{or} \\ k \leq 1, d_\sigma \geq 1-k \quad \forall \sigma \in \Sigma \end{array} \right.$$

where d_F : discriminant of F

w_p : product of "LOCAL GAUSS SUMS" at Σ_p

{ for more general version and details on notation,
refer to the résumé }