p-adic L-functions for CM fields[*1]

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INTRODUCTION OF p-ADIC L-FUNCTIONS FOR CM-FIELDS

- PRELIMINARIES
- PERIODS
- MAIN THEOREM.
- (STRATEGY $\longrightarrow$ $\boxed{\text{DAY 2}}$)

§ PRELIMINARIES

DEFINITION

A number field K is called a CM-FIELD if K is a totally imaginary quadratic extension of a totally real number field F.

$$\begin{array}{cl} K & \\ {\scriptstyle 2}\,| & \mathrm{Gal}(K/F) = \langle c \rangle \leftarrow \text{complex conjugate} \\ \text{totally real}\ \ F & \\ | & \\ \mathbb{Q} & \end{array}$$

- p : prime number
- Fix embeddings $\overline{\mathbb{Q}} \overset{\iota_\infty}{\hookrightarrow} \mathbb{C}$, $\overline{\mathbb{Q}} \overset{\iota_p}{\hookrightarrow} \overline{\mathbb{Q}_p}$
- $I_K := \{\sigma : K \hookrightarrow \mathbb{C}$ embeddings $\}$

  $\cup\!\!|$

  Σ subset

DEFINITION

(K, Σ) : CM-TYPE $\overset{\text{def}}{\Longleftrightarrow}$ (K : CM-field and $I_K = \Sigma \sqcup \Sigma^c$, $\Sigma \cap \Sigma^c = \phi$.)

$(\Sigma^c := \{\sigma \circ c \mid \sigma \in \Sigma\})$

⇨ REMARK.

$$\begin{tikzcd} K \arrow[r, hook, "\sigma"] \arrow[dr, dashed, "\exists!"'] & \mathbb{C} \\ & \overline{\mathbb{Q}} \arrow[u, hook, "\iota_\infty"'] \end{tikzcd}$$

We also denote σ (by abuse of notation)

$\Sigma_p := \{\mathfrak{P} \subseteq \mathcal{O}_K \mid$ prime ideal above p induced by $K \overset{\sigma}{\hookrightarrow} \overline{\mathbb{Q}} \overset{\iota_p}{\hookrightarrow} \overline{\mathbb{Q}}_p \ \ \forall \sigma \in \Sigma\}$

$\Sigma_p^c = \overline{\Sigma}_p := \{\mathfrak{P} \subseteq \mathcal{O}_K \mid$ 〃 $K \overset{\sigma \circ c}{\hookrightarrow} \overline{\mathbb{Q}} \overset{\iota_p}{\hookrightarrow} \overline{\mathbb{Q}}_p \ \ \forall \sigma \in \Sigma\}$

DEFINITION

A CM-type (K, Σ) is <u>p-ORDINARY</u>

$\overset{\text{def}}{\Leftrightarrow}$ $\{$prime ideals of $\mathcal{O}_K$ above $p\} = \Sigma_p \amalg \overline{\Sigma}_p$, $\Sigma_p \cap \overline{\Sigma}_p = \phi$

⇨ **REMARK** K: CM-type.

$$\begin{array}{ccccc} \mathfrak{P} \quad \overline{\mathfrak{P}} & & K & & \sigma \quad \overline{\sigma} \\ \diagdown \ \diagup & & | & & \diagdown \ \diagup \\ \mathfrak{p} & & F & & \nu : \text{real place} \\ | & & | & & \\ p & & \mathbb{Q} & & \end{array}$$

$^{\exists}p$-ordinary CM-type $\Leftrightarrow$ $^{\forall}\mathfrak{p} \subseteq \mathcal{O}_F$ above p splits in K

If exists,

$$\# \left\{ \begin{array}{c} p\text{-ordinary} \\ \text{CM-types} \end{array} \right\} = 2^{\#\left\{ \begin{array}{c} \mathfrak{p} \subseteq \mathcal{O}_F \\ \text{above } p \end{array} \right\}} \leqslant 2^{[F:\mathbb{Q}]} = \#\{\text{CM-types}\}$$

"which $\mathfrak{P}$ you choose for each $\mathfrak{p} \subseteq \mathcal{O}_F$" "which σ you choose for each real place ν of F" ▲

DEFINITION

K: CM field. A character $\lambda : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$ is called

a <u>GRÖSSENCHARACTER OF TYPE (A_0)</u> of infinite type $\kappa = \sum_{\sigma \in I_K} \kappa_\sigma \sigma \in \mathbb{Z}[I_K]$

(or <u>ALGEBRAIC HECKE CHARACTER</u>)

if

$$\lambda_\infty(z_\infty) = \prod_{\substack{\nu : \text{archimedean places} \\ \| \\ \{\sigma_\nu, \overline{\sigma}_\nu\}}} z_\nu^{\kappa_{\sigma_\nu}} \overline{z}_\nu^{\kappa_{\overline{\sigma}_\nu}} \quad \left(=: z_\infty^{\kappa}\right) \quad \forall z \in \mathbb{A}_K^\times.$$

⇨ REMARK.

(1) $\lambda : \mathbb{A}_K^\times / K^\times \to \mathbb{C}$ Größencharacter of type (A_0) of infinite type $\kappa = \sum_{\sigma \in I_K} \kappa_\sigma \sigma$

⇒ Automatically we have $\kappa_{\sigma_v} + \kappa_{\overline{\sigma}_v} \equiv k \in \mathbb{Z}$ (independent of v)

So if we fix a CM-type Σ, we can uniquely rewrite

$$\kappa = \underbrace{k}_{\text{1 variable}} \Sigma + \sum_{\sigma \in \Sigma} \underbrace{d_\sigma}_{[F:\mathbb{Q}]\text{-variable } (\#\Sigma = [F:\mathbb{Q}])} (\sigma - \overline{\sigma}) \qquad k, d_\sigma \in \mathbb{Z} \quad ({}^\forall \sigma \in \Sigma)$$

$$\left(k\Sigma := k \sum_{\sigma \in \Sigma} \sigma \right)$$

⇝ "$([F:\mathbb{Q}]+1)$-variable"

in this case, λ is called "of weight $-k$" (as a pure motif)

(2) $\lambda : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}$ of type (A_0) of infinite type $\kappa = k\Sigma + \sum_{\sigma \in \Sigma} d_\sigma (\sigma - \overline{\sigma})$

$$\Rightarrow \mathrm{Im}\left(\lambda\big|_{\mathbb{A}_{K,fin}^\times}\right) \subseteq {}^\exists L/\mathbb{Q} \quad \text{finite extension.}$$

so we can define:

<u>DEFINITION</u> λ : as above of infinite type $\kappa = k\Sigma + \sum_{d \in \Sigma} d_\sigma (\sigma - \overline{\sigma})$

⇝ define $\widehat{\lambda} : \mathbb{A}_K^\times / K^\times \longrightarrow \overline{\mathbb{Q}}_p^\times$ by

$$\widehat{\lambda}(x) := i_p\Big(\underbrace{\lambda(x)\, x_\infty^{-\kappa}}_{\in \overline{\mathbb{Q}} \text{ by the above remark}}\Big) \times \prod_{\mathfrak{P} \in \Sigma_p} \prod_{\substack{\sigma : K \hookrightarrow \overline{\mathbb{Q}} \\ \sigma_p := \circ\sigma \\ \text{induces } \mathfrak{P}}} \sigma_p(x_{\mathfrak{P}})^{\kappa_\sigma}\, \overline{\sigma}_p(x_{\overline{\mathfrak{P}}})^{\kappa_{\overline{\sigma}}}$$

$\widehat{\lambda}$ is called the <u>p-ADIC AVATAR OF λ</u>

§ PERIODS

(K, Σ) p-ordinary CM-type

$\nu \in I_F := \{\nu : F \hookrightarrow \mathbb{R} \text{ embeddings}\}$

$\downarrow \quad \wr\downarrow$

$\sigma \in \Sigma$

s.t. $\sigma|_F = \nu$

via this, identify

$$\begin{array}{ccc} F \otimes \mathbb{C} & \simeq & \mathbb{C}^{I_F} \\ & & \wr\downarrow \\ K \otimes \mathbb{R} & \simeq & \mathbb{C}^{\Sigma} \end{array}$$

$\boxed{\text{GOAL}}$ $\mathfrak{c} \subseteq F$ fractional ideal prime to p

construct $\mathfrak{c}$-polarized HILBERT-BLUMENTHAL abelian variety of level $\Gamma_{00}(p^\infty)$

$(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K))/_{\mathbb{C}}$ WITH COMPLEX MULTIPLICATION BY $\mathcal{O}_K$

- $X(\mathcal{O}_K) = \mathbb{C}^{I_F}/\mathcal{O}_K \circlearrowleft \mathcal{O}_K$ [via above identification, $\mathcal{O}_K$ can be regarded as a lattice of $\mathbb{C}^{I_F}$]

  : complex torus

- $\mathfrak{c}$-polarization : i.e., $\wedge^2_{\mathcal{O}_F}(\mathcal{O}_K) \xrightarrow{\sim} \mathfrak{c}^{-1}\mathfrak{d}_F^{-1}$ ($\mathfrak{d}_F$: different of F)

(∗). Choose $\delta \in K$ purely imaginary (i.e., $\bar{\delta} = -\delta$) s.t.,

- the image of $\delta \otimes 1 \in K \otimes \mathbb{R} \xrightarrow{\sim} F \otimes \mathbb{C}$ is contained in $F \otimes \mathbb{R}$ and totally positive (in $F \otimes \mathbb{R}$)
- alternating pairing on $\mathcal{O}_K$ $\langle u, v \rangle_\delta := \dfrac{\bar{u}v - u\bar{v}}{2\delta}$ defines an isomorphism of invertible $\mathcal{O}_F$-modules : $\wedge^2_{\mathcal{O}_F}(\mathcal{O}_K) \xrightarrow[\sim]{\langle\,,\,\rangle_\delta} \mathfrak{c}^{-1}\mathfrak{d}_F^{-1}$.

$\rightsquigarrow \left(X(\mathcal{O}_K), \langle\ ,\ \rangle_\delta\right)$ ($\mathfrak{c}$-polarized complex torus) defines

a $\mathfrak{c}$-polarized HBAV $\left(X(\mathcal{O}_K), \lambda(\delta)\right)_{/\mathbb{C}}$

$$\lambda(\delta) : X(\mathcal{O}_K) \longrightarrow X(\mathcal{O}_K)^t \underset{\mathcal{O}_F}{\otimes} \mathfrak{c}$$

• $\Gamma_{00}(p^\infty)$-level structure : i.e., $\mathfrak{d}_F^{-1} \otimes \varprojlim_n \mu_{p^n} \hookrightarrow \mathcal{O}_K \otimes \mathbb{Z}_p$.

Set $\varphi_\Sigma : \mathcal{O}_K \otimes \mathbb{Z}_p \longrightarrow \prod_{\mathfrak{P} \in \Sigma_p} \widehat{\mathcal{O}}_{K,\mathfrak{P}} \xrightarrow{\sim} \prod_{\substack{\mathfrak{p} \in \mathcal{O}_F \\ \text{above } p}} \widehat{\mathcal{O}}_{F,\mathfrak{p}} = \mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$

◎ $\mathcal{O}_K \otimes \mathbb{Z}_p \xrightarrow{\sim} (\mathcal{O}_F \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p)$.

$$\begin{array}{ccc} \rightsquigarrow \quad \mathcal{O}_K \otimes \mathbb{Z}_p & & (\mathcal{O}_F \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \\ \Psi & & \Psi \\ x & \longmapsto & \left(\varphi_\Sigma(x),\ \varphi_\Sigma(\bar{x})\right) \end{array}$$

◎ $\boxed{\mathcal{O}_F \otimes \mathbb{Z}_p \overset{(*)}{\simeq} \mathfrak{d}_F^{-1} \otimes \mathbb{Z}_p}$

$\because$) Since $\mathcal{O}_K \otimes \mathbb{Z}_p \simeq (\mathcal{O}_F \otimes \mathbb{Z}_p)^2$

$$\begin{array}{ccc} \Lambda^2_{\mathcal{O}_F \otimes \mathbb{Z}_p}(\mathcal{O}_K \otimes \mathbb{Z}_p) & \xrightarrow{\sim} & \mathcal{O}_F \otimes \mathbb{Z}_p. \\ & \searrow^{\sim}_{\langle\ ,\ \rangle_\delta} & \\ & & \mathfrak{d}_F^{-1}\mathfrak{c}^{-1} \otimes \mathbb{Z}_p = \mathfrak{d}_F^{-1} \otimes \mathbb{Z}_p \end{array}$$

$\mathfrak{c}$: prime to p ■

◎ $\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} \xrightarrow[\exp]{\sim} \varprojlim_n \mu_{p^n}$; $(a_n)_{n \geq 1} \longmapsto \left(\exp\left(2\pi\sqrt{-1} \cdot \frac{a_n}{p^n}\right)\right)_{n \geq 1}$.

Combining them,

$$\begin{tikzcd} (\mathcal{O}_F \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \arrow[d, "\wr", "(*)\text{ for first component}"'] & \mathcal{O}_K \otimes \mathbb{Z}_p \arrow[l, "(\varphi_\Sigma(x),\, \varphi_\Sigma(\bar{x}))"'] \\ (\mathfrak{d}_F^{-1} \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p) & \\ \mathfrak{d}_F^{-1} \otimes \mathbb{Z}_p \arrow[u, hook] \arrow[d, "\wr", "\exp"'] & \\ \mathfrak{d}_F^{-1} \otimes \varprojlim \mu_{p^n} \arrow[uuur] & \end{tikzcd}$$

$\boxed{i(\mathcal{O}_K) : \mathfrak{d}_F^{-1} \otimes \varprojlim_n \mu_{p^n} \hookrightarrow \mathcal{O}_K \otimes \mathbb{Z}_p}$

$\Gamma_{00}(p^\infty)$-level structure.

So we have constructed $(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K))_{/\mathbb{C}}$

$\xrightarrow{\text{THEORY OF COMPLEX MULTIPLICATION}}$ • $(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K))$ is defined over $\overline{\mathbb{Q}}$

• at the prime of $\mathcal{O}_{\overline{\mathbb{Q}}}$ induced by $\mathcal{O}_{\overline{\mathbb{Q}}} \hookrightarrow \overline{\mathbb{Q}}_p$,

$(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K))$ has a <u>good ordinary reduction</u>.

i.e., has a model over

$$A = \{ x \in \overline{\mathbb{Q}} \mid \iota_p(x) \in \mathcal{D}_p \}.$$

$\mathcal{D}_p$: valuation ring of $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$

$$\begin{tikzcd}
(X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K))_{/\mathbb{C}} & & (X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K))_{/\mathcal{D}_p} \\
& (X(\mathcal{O}_K), \lambda(\delta), i(\mathcal{O}_K))_{/A} \arrow[ul, squiggly, "\iota_\infty"] \arrow[ur, squiggly, "\iota_p"'] &
\end{tikzcd}$$

$\exists \omega_{trans}$ $\qquad\qquad$ $\exists \omega_{can}$

$\mathbb{C}^\Sigma \cong \mathbb{C}^{I_F} \twoheadrightarrow X(\mathcal{O}_K)$

$(dz_\sigma)_{\sigma\in\Sigma} \longleftrightarrow \omega_{trans}$

choose $\omega \in \underline{\omega}_{X(\mathcal{O}_K)/A}$ — locally free $\mathcal{O}_F \otimes A$-module of rank 1

$\qquad = H^0(X(\mathcal{O}_K), \Omega_{X(\mathcal{O}_K)/A})$ "invariant 1-form"

DEFINITION

- on $\mathbb{C}$, $\quad \omega = {}^\exists\Omega_{CM}\, \omega_{trans} \quad \Omega_{CM} = (\Omega_{CM}(\sigma))_{\sigma\in\Sigma} \in (\mathcal{O}_F \otimes \mathbb{C})^\times \simeq (\mathbb{C}^\times)^\Sigma$. CM-PERIOD
- on $\mathcal{D}_p$, $\quad \omega = {}^\exists\Omega_p\, \omega_{can} \quad \Omega_p = (\Omega_p(\sigma))_{\sigma\in\Sigma} \in (\mathcal{O}_F \otimes \mathcal{D}_p)^\times \simeq (\mathcal{D}_p^\times)^\Sigma$ p-ADIC PERIOD

☞ REMARK.

each Ω_{CM}, Ω_p DOES depend on the choice of ω,

BUT the "ratio" of (Ω_{CM}, Ω_p) is independent of ω

☞ SUPPLIMENTARY COMMENTS (definitions of ω_{trans}, ω_{can})

By the duality $\underline{\omega}_{X(O_K)/R} \otimes_{\mathbb{Q}\otimes R} Lie(X(O_K)/R) \xrightarrow{\sim} \mathfrak{d}_F^{-1} \otimes_{\mathbb{Z}} R$,

to take an invariant 1-form $\omega \in \underline{\omega}_{X(O_K)/R}$ is equivalent to determine

an isomorphism. $Lie(X(O_K)/R) \xrightarrow{\sim} \mathfrak{d}_F \otimes_{\mathbb{Z}} R$.

1° ω_{trans} ($R=\mathbb{C}$ case)

By construction, $Lie(X(O_K)/\mathbb{C}) = Lie(X(O_K)^{an}) = Lie(\underset{\underset{F\otimes\mathbb{C}}{\cong}}{\mathbb{C}^{I_F}}/O_K)$

$= F\otimes\mathbb{C} = \mathfrak{d}^{-1}\otimes\mathbb{C}$

ω_{trans} is induced by this isomorphism (equality).

2° ω_{can} ($R=\mathcal{D}_p$ case)

The level structure $i(O_K)$ induces $\mathfrak{d}_F^{-1}\otimes\widehat{\mathbb{G}}_m \xrightarrow{\sim} \widehat{X}$

as formal schemes / $\mathcal{D}_p$.

This induces on Lie groups

$$\begin{array}{ccc} \mathfrak{d}_F^{-1}\otimes Lie(\widehat{\mathbb{G}}_m/\mathcal{D}_p) & \xrightarrow{\sim} & Lie(\widehat{X}/\mathcal{D}_p) \\ \| & & \| \\ \mathfrak{d}_F^{-1}\otimes\mathcal{D}_p & & Lie(X/\mathcal{D}_p) \end{array}$$

ω_{can} is induced by the (inverse of the) left isomorphism

§ MAIN THEOREM ([K78, THEOREM 5.30] [HT93, THEOREM II, THEOREM 4.1])

(K, Σ) : p-ordinary CM-type.

choose $\delta \in K$ satisfying (*)

$\Rightarrow \exists !$ $\mathcal{D}_p$-valued measure μ on $Cl_K(p^\infty)$
(ray class group modulo p^∞)

s.t.,

$$\frac{\int_{Cl_K(p^\infty)} \hat{\lambda}\, d\mu}{\Omega_p^{k\Sigma+2d}} = [\mathcal{O}_K^\times : \mathcal{O}_F^\times]\, w_p(\lambda) \frac{(-1)^{k[F:\mathbb{Q}]}\, \pi^{|d|}}{\sqrt{d_F}\; Im(\delta)^{d}} \times \frac{\Gamma_\Sigma(k\Sigma + d)}{\Omega_{CM}^{k+2d}}$$

$$\times \prod_{\mathfrak{P} \in \Sigma_p} (1 - \lambda(\bar{\mathfrak{P}}))(1 - \check{\lambda}(\bar{\mathfrak{P}}))\; L(\lambda, 0)$$

for all Größencharacter of type (A_0) λ whose conductor dividing p^∞

of type $\kappa = k\Sigma + \sum_{\sigma \in \Sigma} d_\sigma (\sigma - \bar{\sigma})$ satisfying

$$(\#) \begin{cases} k \geq 1, \ d_\sigma \geq 0 & \forall \sigma \in \Sigma \\ \text{or} & \\ k \leq 1, \ d_\sigma \geq 1 - k & \forall \sigma \in \Sigma \end{cases}$$

where d_F : discriminant of F

w_p : product of "LOCAL GAUSS SUMS" at Σ_p.

[for more general version and details on notation, refer to the résumé]