Construction of p-adic L-functions for CM-fields à la Katz, Hida, Tilouine

- Strategy
- Hecke's Eisenstein series.
- Katz' Eisenstein measure and p-adic L-functions
- (· Arithmetic differential operators $\longrightarrow$ Day 3)

§ Strategy.

· Constructing "Eisenstein measure"

$\underbrace{V(\mathfrak{c}\,;\,\Gamma_{00}(p^\infty))}$ - valued measure.

space of p-adic $\mathfrak{c}$-HMF of level $\Gamma_{00}(p^\infty)$ (recall Nuccio's talk)

$\rightsquigarrow$ "(p-adic L-function) = $\sum$ (evaluation of Eisenstein measure at "CM points")"

$\updownarrow$ moduli interpretation

$\mathfrak{c}$-HBAV with CM

__Notation__ (review)

p : prime number $\qquad (K, \Sigma)$: p-ordinary CM-type
$\quad\cup\!\mid$
$\quad F$: maximal totally real subfield

$\mathfrak{c} \subseteq F$ fractional ideal, prime to p (polarization ideal)

§ HECKE'S EISENSTEIN SERIES

$$\phi : (\mathcal{O}_F \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \longrightarrow \mathbb{C} \quad \text{locally constant function}$$

PARTIAL FOURIER TRANSFORM

$$\phi : \text{constant on } p^n\big((\mathcal{O}_F \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p)\big) \longrightarrow \mathbb{C}$$

$$\rightsquigarrow \mathcal{P}\phi : \left(\mathfrak{d}_F^{-1} \otimes \tfrac{1}{p^n}\mathbb{Z}_p/\mathbb{Z}_p\right) \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \longrightarrow \mathbb{C}$$

defined by $\displaystyle \mathcal{P}\phi(x, y) := \frac{1}{p^{n[F:\mathbb{Q}]}} \sum_{u \in (\mathcal{O}_F/p^n\mathcal{O}_F)} \phi(u, y) \exp(2\pi\sqrt{-1}\, x u)$

☞ SUPPLIMENTARY COMMENTS

$\mathcal{P}\phi$ DOES NOT depend on the choice of n as above, so we regard $\mathcal{P}\phi$ as an function on $(\mathfrak{d}_F^{-1} \otimes \mathbb{Q}_p/\mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p)$ by zero-extension ◢

"PARTIAL" TATE MODULES

$(\mathcal{L}, \langle\ ,\ \rangle, i)$: τ-polarized lattice in $\mathbb{C}^{I_F} \simeq F \otimes \mathbb{C}$

$\rightsquigarrow (X(\mathcal{L}), \lambda(\mathcal{L}), \omega_{\mathrm{trans}}(\mathcal{L}), i(\mathcal{L}))_{/\mathbb{C}}$ $\quad \tau$-HBAV$_{/\mathbb{C}}$

[similarly to the recipe of construction of $(X(\mathcal{O}_K), \lambda(\delta), \omega_{\mathrm{trans}}, i(\mathcal{O}_K))$]

$$(b)\ 0 \to \mathfrak{d}_F^{-1}\otimes \mathbb{Z}_p \xrightarrow{\ i\ } \mathcal{L}\otimes \mathbb{Z}_p \longrightarrow \oslash \longrightarrow 0.$$

[CAUTION: $\mathfrak{d}_F^{-1}$ coincides with ∂ in Nuccio's talk: he defines ∂ as $\mathrm{Diff}(F/\mathbb{Q})^{-1}$]

As yesterday,

by the above exact sequence.

$$\Lambda^2_{\mathcal{O}_F\otimes\mathbb{Z}_p}(\mathcal{L}\otimes\mathbb{Z}_p) \simeq (\mathfrak{d}_F^{-1}\otimes\mathbb{Z}_p)\otimes_{\mathcal{O}_F\otimes\mathbb{Z}_p} \oslash$$

$$\xrightarrow[\sim]{\langle\,,\,\rangle} \mathfrak{d}_F^{-1}\tau^{-1}\otimes\mathbb{Z}_p = \mathfrak{d}_F^{-1}\otimes\mathbb{Z}_p$$

τ: prime to p

$$\rightsquigarrow \boxed{\oslash \cong \mathcal{O}_F\otimes\mathbb{Z}_p}$$

$$(b)\otimes\mathbb{Q}_p \rightsquigarrow 0 \to \mathfrak{d}_F^{-1}\otimes\mathbb{Q}_p \xrightarrow{\ i\ } \mathcal{L}\otimes\mathbb{Q}_p \to \mathcal{O}_F\otimes\mathbb{Q}_p \longrightarrow 0$$

DEFINITION

$$PV_p\mathcal{L} := \mathcal{L}\otimes\mathbb{Z}_p + i(\mathfrak{d}_F^{-1}\otimes\mathbb{Q}_p) \subseteq \mathcal{L}\otimes\mathbb{Q}_p$$

By construction;

$$0 \to \mathcal{L}\otimes\mathbb{Z}_p \to PV_p\mathcal{L} \xrightarrow{pr_1} \mathfrak{d}_F^{-1}\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

$$0 \to \mathfrak{d}_F^{-1}\otimes\mathbb{Q}_p \to PV_p\mathcal{L} \xrightarrow{pr_2} \mathcal{O}_F\otimes\mathbb{Z}_p \longrightarrow 0$$

For $\phi : (\mathcal{O}_F\otimes\mathbb{Z}_p)\times(\mathcal{O}_F\otimes\mathbb{Z}_p) \longrightarrow \mathbb{C}$ locally constant

and $l \in PV_p\mathcal{L}$, define

$$P\phi(l) := P\phi(pr_1(l), pr_2(l))$$

<u>THEOREM</u> (HECKE, [K78, THEOREM (3.2.3)])

for $k \geq 1$, $\phi : (\mathcal{O}_F \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \longrightarrow \mathbb{C}$ locally constant

s.t., $\begin{cases} \cdot \ \mathrm{Supp}(\phi) \subseteq (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \\ \cdot \ \forall e \in \mathcal{O}_F^\times, \quad \phi(e^{-1}x, ey) = N_{F/\mathbb{Q}}(e)^k \phi(x, y). \end{cases}$

Then

$$G_{k,\phi}((\mathcal{L}, \langle\,,\,\rangle, i(\mathcal{L})); s)$$

$$:= \frac{(-1)^{k[F:\mathbb{Q}]}\,\Gamma(k+s)^{[F:\mathbb{Q}]}}{\sqrt{d_F}} \sum_{\substack{l \in \mathcal{L}[1/p] \cap PV_p\mathcal{L} \\ \mathrm{mod}\ \mathcal{O}_F^\times}} \frac{P\phi(l)}{N_{F/\mathbb{Q}}(l)^k \,|N_{F/\mathbb{Q}}\, l|^{2s}}$$

has analytic continuation on whole $\mathbb{C}$ (w.r.t. s)

Set $G_{k,\phi} := G_{k,\phi}(s)|_{s=0}$ ... $\mathfrak{c}$-HMF of (parallel) weight k of level $\Gamma_{00}(p^\infty)$.

$G_{k,\phi}$ has a q-expansion at a cusp determined by $(\mathfrak{a}, \mathfrak{b})$ ($\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$ $\mathfrak{a}, \mathfrak{b} \subseteq F$

given by $\mathfrak{a}$: prime to p)

$$G_{k,\phi}(q; \mathfrak{a}, \mathfrak{b}) = N\mathfrak{a} \sum_{\substack{\alpha \in \mathfrak{a}\mathfrak{b} \\ \alpha \gg 0}} q^\alpha \sum_{\substack{\alpha = ab \\ a \in \mathfrak{a},\, b \in \mathfrak{b} \\ \mathrm{modulo}\ \mathcal{O}_E^\times}} \underbrace{\mathrm{sgn}(N_{F/\mathbb{Q}}a)\, N_{F/\mathbb{Q}}(a)^{k-1}}_{\in\, \mathbb{Z}_{(p)}\ (\mathfrak{a}:\ \text{prime to } p)} \phi(a, b)$$

(for details, see résume)

◎ ϕ : $\mathbb{Z}_p$-valued, continuous $\Rightarrow$ $\phi \bmod p^n : (O_F \otimes \mathbb{Z}_p)^2 \to \mathbb{Z}/p^n\mathbb{Z}$ locally constant

$\Rightarrow G_{k,\phi} \bmod p^n \in M_{k\Sigma}(\tau ; \mathbb{Z}/p^n\mathbb{Z})$

by q-expansion principle

(COROLLARY TO RIBET'S THEOREM in NUCCIO'S talk)

As in NUCCIO'S talk, "compatible system" of classical τ-HMF's $\{G_{k,\phi} \bmod p^n\}_{n \geq 1}$ determines a p-adic modular form

$$G_{k,\phi}^{(p)} \in V(\tau, \Gamma_{00}(p^\infty) ; \mathbb{Z}_p)$$

[This construction is also valid for any p-adic ring $R = \varprojlim_n R/p^nR$ via "basechange"]

☞ SUPPLIMENTARY COMMENTS

It is a quite classical observation (by HECKE etc...) that the evaluation of such EISENSTEIN series "at CM-point" gives the value of L-functions:

TYPICAL EXAMPLE : (rough)

$$\text{“} \sum \frac{1}{(az+b)^k} \bigg|_{z=\sqrt{-1}} = \sum \frac{1}{(a\sqrt{-1}+b)^k} = \zeta_{\mathbb{Q}(\sqrt{-1})}(k) \text{”}$$

and $\sqrt{-1} \in \mathfrak{h}$ is a CM point on the modular curve.

Such observation might motivate KATZ to consider

"evaluation of p-adic EISENSTEIN series at CM POINTS also gives L-VALUES"

[KENICHI NAMIKAWA (NATIONAL TAIWAN UNIVERSITY) suggested that
OSAKA →
such typical examples should help one to understand the essential idea of KATZ' construction, and hence I added this comment here.
I am very grateful to NAMIKAWA-sun for his valuable suggestion:].

§ KATZ' EISENSTEIN MEASURE AND p-ADIC L-FUNCTIONS

Let $\phi : (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \longrightarrow R$. p-adic ring (i.e., $R = \varprojlim R/p^n R$)

s.t., $\phi(ex, ey) = \phi(x, y) \quad {}^\forall e \in \mathcal{O}_F^\times$

Set $\phi' : (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \longrightarrow R$

by $\phi'(x, y) := \dfrac{1}{N_{F/\mathbb{Q}}(x)} \phi\left(\dfrac{1}{x}, y\right)$

⇨ REMARK, $\phi'(e^{-1}x, ey) = N_{F/\mathbb{Q}}(e)\,\phi'(x, y) \quad {}^\forall e \in \mathcal{O}_F^\times$

DEFINITION (KATZ' EISENSTEIN MEASURE)

$G := (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times / \overline{\mathcal{O}_F^\times}$ ← closure $\quad (\mathcal{O}_F^\times \overset{\text{diagonal}}{\hookrightarrow} (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times)$

$\rightsquigarrow \mathbb{E}_\tau : \mathrm{Cont}(G, R) \longrightarrow V(\tau, \Gamma_{00}(p^\infty); R) ; \quad \int_G \phi\, d\mathbb{E}_\tau = G^{(p)}_{1,\phi'}$

⇨ REMARK Since $\|f\|_p := \sup_x |f(x)|_p$ is bounded in $V(\tau, \Gamma_{00}(p^\infty); \underset{\text{integral}}{R})$, the distribution $\mathbb{E}_\tau$ becomes "automatically" a <u>measure</u>

CONSTRUCTION OF p-ADIC L-FUNCTION

by class field theory,

closure

$$\exists\ 0 \to (O_K \otimes \mathbb{Z}_p)^\times / \overline{O_K}^\times \to Cl_K(p^\infty) \to Cl_K \to 0$$

$$(O_K \otimes \mathbb{Z}_p)^\times / \overline{O_K}^\times \cong G_0$$

ideal class group

$$(O_K \otimes \mathbb{Z}_p)^\times / \overline{O_K}^\times \xrightarrow[\cong]{(\varphi_\Sigma(x),\, \varphi_\Sigma(\bar{x}))} (O_F \otimes \mathbb{Z}_p)^\times \times (O_F \otimes \mathbb{Z}_p)^\times / \overline{O_K}^\times \hookleftarrow G$$

Choose representative $\{\mathfrak{A}_j\}_{j=1}^{h}$ of Cl_K

as $\mathfrak{A}_j \subseteq O_K$, prime to p

$$\rightsquigarrow Cl_K(p^\infty) = \coprod_{j=1}^{\# Cl_K} [\mathfrak{A}_j]^{-1} G_0.$$

each $\mathfrak{A}_j \rightsquigarrow (X(\mathfrak{A}_j), \lambda(\delta), i(\mathfrak{A}_j))/A$.

(RECALL $A = \{x \in \overline{\mathbb{Q}} \mid i_p(x) \in \mathcal{D}_p\}$ $\mathcal{D}_p$: valuation ring of $\mathbb{C}_p$)

NOTE: $\mathfrak{A}_j \otimes \mathbb{R} = O_K \otimes \mathbb{R}$

$\mathfrak{A}_j \otimes \mathbb{Z}_p = O_K \otimes \mathbb{Z}_p$ since $\mathfrak{A}_j$: prime to p.

Especially, we can take the same δ for each $\mathfrak{A}_j$

and if choose one $\omega \in \underline{\omega}_{X(O_K)/A}$, by taking

$\omega_{\mathfrak{A}_j}: Lie(X(\mathfrak{A}_j)/A) = Lie(X(O_K)/A) \xrightarrow{\sim} \delta^{-1} \otimes A$ simultaneously, we can use the same periods (Ω_{CM}, Ω_p) for each $X(\mathfrak{A}_j)$

DEFINITION

Set $\mathbb{E}(\mathfrak{A}_j) := \mathbb{E}_{\kappa N_{K/F}(\mathfrak{A}_j)^{-1}}\big(\underbrace{(X(\mathfrak{A}_j), \lambda(\delta), i(\mathfrak{A}_j))/\mathcal{D}_p}_{\text{CM point}}\big)$ ($\mathcal{D}_p$-valued measure)

Then define.

$$\mu : \mathrm{Cont}(Cl_K(p^\infty), \mathcal{D}_p) \longrightarrow \mathcal{D}_p$$

$$f \longmapsto \sum_{j=1}^{\# Cl_K} \int_G f([\mathfrak{A}_j]^{-1} g)\, d\mathbb{E}(\mathfrak{A}_j).$$

[μ "should be" the desired p-adic L-function]

◎ COMPARISON v.s. DELIGNE - RIBET CONSTRUCTION.

	DELIGNE - RIBET	KATZ
looking at	values at cusps (constant terms)	values at CM points
construction of measures.	"complicated" we need congruences between L-values	"simple" "easy(?)" at least, we $\boxed{\text{DO NOT}}$ need $\boxed{\text{any}}$ congruences between L-values

So what is difficult in KATZ' construction?

DIFFICULTY : How to compute "evaluation at CM points" of p-adic EISENSTEIN series ?

We only know description at cusps (!!)
(i.e., q-expansions)

NAIVE OBSERVATION : λ : Größencharacter of type (A_0) of infinite type $\varkappa = k\Sigma + \sum_{\sigma \in \Sigma} d_\sigma(\sigma - \bar{\sigma})$

$$\int_{Cl_K(p^\infty)} \hat{\lambda}\, d\mu \overset{\text{by definition}}{=} \sum_{j=1}^{\# Cl_K} \int_G \underbrace{\hat{\lambda}([\mathfrak{A}_j]^{-1} g)}_{\lambda(\mathfrak{A}_j)^{-1}\hat{\lambda}(g)} \, d\mathbb{E}(\mathfrak{A}_j)$$

$$= \sum_{j=1}^{\# Cl_K} \lambda(\mathfrak{A}_j)^{-1} \underbrace{\int_G \hat{\lambda}(g)\, d\mathbb{E}(\mathfrak{A}_j)}_{(*)}$$

Note that $\hat{\lambda}(g) = \lambda_{\Sigma_p}(x)^{-1} \lambda_{\bar{\Sigma}_p}(y)^{-1} \cdot \dfrac{y^{d}}{x^{k\Sigma + d}}$

if we write "p-component of g" by $g_p = (x, y) \in (K \otimes \mathbb{Z}_p)^\times \xrightarrow[(\varphi_\Sigma(\alpha), \varphi_{\bar\Sigma}(\alpha))]{\sim} (F \otimes \mathbb{Z}_p)^{\times, 2}$

Hence we can compute (recall the definition of $\mathbb{E}_{\varepsilon N_{F/K}(\mathfrak{A}_j)^{-1}}$ and ϕ')

$$(*) = G_{1, N(x)^{-1}\lambda_{\Sigma_p}(x)\lambda_{\bar{\Sigma}_p}(y)^{-1} x^{k\Sigma + d} y^{d}}\left((X(\mathfrak{A}_j), \lambda(\delta), i(\mathfrak{A}_j)) / \mathcal{D}_p \right)$$

$$= G_{1, N(x)^{k-1}\lambda_{\Sigma_p}(x)\lambda_{\bar{\Sigma}_p}(y)^{-1} (xy)^{d}}\left((X(\mathfrak{A}_j), \lambda(\delta), i(\mathfrak{A}_j)) / \mathcal{D}_p \right)$$

[Sorry for double-notation on λ

$N_{F/\mathbb{Q}}(x) = \prod_{\sigma \in \Sigma} x^\sigma$ (recall the identification $I_F \simeq \Sigma$)

◎ Observe q-expansion at the cusp $(\mathfrak{a}, \mathfrak{b})$ ($\mathfrak{a}$: prime to p).

$$G_{1, N(x)^{k-1}\lambda_{\Sigma_p}(x)\lambda_{\bar{\Sigma}_p}(y)^{-1}(xy)^{d}}(q; \mathfrak{a}, \mathfrak{b})$$

how to deal with?

$$= N\mathfrak{a} \sum_{\substack{\alpha \in \mathfrak{a}\mathfrak{b} \\ \alpha \gg 0}} q^{\alpha} \sum_{\substack{\alpha = ab \\ a \in \mathfrak{a}, b \in \mathfrak{b} \\ \text{modulo } \mathcal{O}_F^{\times}}} \operatorname{sgn}(N_{F/\mathbb{Q}}a) N_{F/\mathbb{Q}}(a)^{k-1} \underbrace{\lambda_{\Sigma_p}(a)\lambda_{\bar{\Sigma}_p}(b)^{-1}}_{\text{locally constant}} \cdot \boxed{(ab)^{d}} \quad \cdots\cdots (\ast\ast)$$

define (formally here) differential operator $\theta(\nu)$ $\nu \in I_F$.

as. $$\theta(\nu)\left(\sum_{\alpha \in \mathfrak{a}\mathfrak{b}} a_\alpha q^\alpha\right) = \sum_{\alpha \in \mathfrak{a}\mathfrak{b}} \nu(\alpha) a_\alpha q^\alpha$$

for $k = \sum_{\nu \in I_F} k_\nu \nu$. $\theta(k) := \prod_{\nu \in I_F} \theta(\nu)^{k_\nu}$

$$\rightsquigarrow (\ast\ast) = \theta(d)\left(N\mathfrak{a} \sum_{\substack{\alpha \in \mathfrak{a}\mathfrak{b} \\ \alpha \gg 0}} q^{\alpha} \sum_{\substack{\alpha = ab \\ a \in \mathfrak{a}, b \in \mathfrak{b}. \\ \text{modulo } \mathcal{O}_F^{\times}}} \operatorname{sgn}(N_{F/\mathbb{Q}}a) N_{F/\mathbb{Q}}(a)^{k-1} \lambda_{\Sigma_p}(a)\lambda_{\bar{\Sigma}_p}(b)^{-1}\right)$$

$$= \theta(d)\left(\underline{G_{k, \lambda_{\Sigma_p}(x)\lambda_{\bar{\Sigma}_p}(y)^{-1}}}\right)$$

(by q-expansion principle)

$\lambda_{\Sigma_p}(x)\lambda_{\bar{\Sigma}_p}(y)^{-1}$: locally constant

$\rightsquigarrow$ can compute by using DIRICHLET series expression.

$\boxed{\text{KEY IDEA}}$ (KATZ' "L'EXÉCUTION TRANSCENDANTE")

If $\theta(d)$ comes from <u>a differential operator defined over A</u>

$\rightsquigarrow$ we can calculate $\theta(d)\, G_{k,\lambda_{\Xi_p}(x)\lambda_{\Xi_p}(y)^{-1}}$ via analytic computation. ("transcendental")

here $\boxed{\text{we can evaluate at CM points}}$

(we know the <u>analytic</u> expression of

$G_{k,\lambda_{\Xi_p}(x)\lambda_{\Xi_p}(y)^{-1}}$ at CM points (!!))

$$
\begin{tikzcd}
\overset{\theta(d)}{\mathcal{D}_p} & & \overset{\theta(d)_{/\mathbb{C}}}{\mathbb{C}} \\
& \overset{\theta(d)_{/A}}{(X(\mathfrak{A}_j),\ \lambda(\delta),\ i(\mathfrak{A}_j))_{/A}} \arrow[ul, "l_p"] \arrow[ur, "l_\infty"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\theta(d)\, G_{k,\lambda_{\Xi_p}(x)\lambda_{\Xi_p}(y)^{-1}} \arrow[rr, bend right] & & \theta(d)_{\mathbb{C}}\, G_{k,\lambda_{\Xi_p}(x)\lambda_{\Xi_p}(y)^{-1}}.
\end{tikzcd}
$$