CONSTRUCTION OF p-ADIC L-FUNCTIONS FOR CM-FIELDS À LA KATZ, HIDA, TILOUINE (CONTINUED)

- REVIEW
- ARITHMETIC DIFFERENTIAL OPERATORS
- SPLITTINGS OF HODGE FILTRATIONS

(· "BULLET TOUR" AROUND MAIN CONJECTURES
→ Sorry, but "TOUR CANCELLED")

NOTATION

- p : prime number. (K, Σ) : p-ordinary CM-type
$\cup\!\mid$
F : maximal totally real subfield

$\mathfrak{c} \subseteq F$: fractional ideal prime to p (polarization ideal)

§ REVIEW.

Yesterday, we constructed the EISENSTEIN MEASURE.

$$\mathbb{E}_{\mathfrak{c}} : \mathrm{Cont}(G, \mathcal{R}) \longrightarrow V(\mathfrak{c}, \Gamma_{00}(p^\infty); \mathcal{R})$$
$$\phi \longmapsto G_{1,\phi'}^{(p)}$$

$$\left(\begin{array}{l} G := (\mathcal{O}_F \otimes \mathbb{Z}_p)^{\times, 2} / \overline{\mathcal{O}_F^{\times}} \quad \text{closure.} \\ \mathcal{R} = \varprojlim_n \mathcal{R}/p^n\mathcal{R} \end{array} \right)$$

From class field theory.

$$G \longrightarrow Cl_K(p^\infty) \longrightarrow Cl_K \longrightarrow 0$$

$\wr$

$\{\mathfrak{A}_j^{-1}\}_{j=1}^{h}$ representative of Cl_K

$\mathfrak{A}_j \subseteq \mathcal{O}_K$, prime to p.

$$\rightsquigarrow \quad \mu : \mathrm{Cont}(Cl_K(p^\infty), \mathcal{D}_p) \longrightarrow \mathcal{D}_p.$$

$$f \longmapsto \sum_{j=1}^{h} \int_G f([\mathfrak{A}_j]^{-1} g)\, d\mathbb{E}(\mathfrak{A}_j)$$

where

$$\mathbb{E}(\mathfrak{A}_j) := \mathbb{E}_{\mathfrak{c}N_{K/F}(\mathfrak{A}_j)^{-1}}\left((\underline{X}(\mathfrak{A}_j), \lambda(\delta), i(\mathfrak{A}_j))_{/\mathcal{D}_p} \right)$$

Let λ : Größencharacter of type (A_0) of infinite type κ

$$\kappa = k\Sigma + \sum_{\sigma\in\Sigma} d_\sigma(\sigma - \bar{\sigma}) \qquad d := (d_\sigma)_{\sigma\in\Sigma}$$

By definition.

$$\int_{Cl_K(p^\infty)} \hat{\lambda}\, d\mu = \sum_{j=1}^{h} \lambda(\mathfrak{A}_j)^{-1} \underline{G^{(p)}_{1, N(a)^{k-1}\lambda_{\Sigma_p}(a)\lambda_{\bar{\Sigma}_p}(b)^{-1}(ab)^{d}}} \left((\underline{X}(\mathfrak{A}_j), \lambda(\delta), i(\mathfrak{A}_j))_{/\mathcal{D}_p} \right)$$

$\|$ yesterday's observation + q-expansion principle

$$\theta(d)\, G^{(p)}_{k, \lambda_{\Sigma_p}(a)\lambda_{\bar{\Sigma}_p}(b)^{-1}}$$

$\uparrow$ has analytic description.

where for $\nu \in I_F$ $(\simeq \Sigma)$

$$\theta(\nu)\left(\sum_{\alpha\in\mathfrak{ab}} a_\alpha q^\alpha\right) = \sum_{\alpha\in\mathfrak{ab}} \nu(\alpha) a_\alpha q^\alpha \quad \text{differential operator}$$

$$\begin{tikzcd} \mathbb{C} & & \mathcal{D}_p \\ & (X(\mathfrak{A}_j), \lambda(\mathfrak{s}), i(\mathfrak{A}_j))_{/A} \arrow[ul, squiggly, "i_\infty"'] \arrow[ur, squiggly, "i_p"] \arrow[ul, bend left=40, "\theta(d)"] \arrow[ur, bend right=40, "\theta(d)"'] & \end{tikzcd}$$

$$\boxed{\begin{array}{l} \underline{\text{KATZ' "L'EXÉCUTION TRANSCENDANTE"}} \\ \text{If } \theta(d) \text{ comes from a differential operator}_{/\underset{\sim}{A}}, \\ \text{we can compute "evaluation at CM-points"} \\ \text{by using "archimedean description,"} \\ \qquad \text{(i.e., DIRICHLET SERIES expression)} \end{array}}$$

§ ARITHMETIC DIFFERENTIAL OPERATORS. ([KAT 78, CHAPTER II])

$\mathcal{M} := \mathcal{M}(\mathfrak{r}, \Gamma_{00}(p^\infty))_{/A}$ moduli of $\mathfrak{r}$-HBAV of level $\Gamma_{00}(p^\infty)$.

$(\pi : \mathfrak{X}_{univ} \to \mathcal{M}, \lambda_{univ}, i_{univ})$: universal $\mathfrak{r}$-HBAV of level $\Gamma_{00}(p^\infty)$

Set $\underline{\omega} := \pi_* \Omega^1_{\mathfrak{X}_{univ}/\mathcal{M}}$, $\underline{H}^1_{DR} := \mathbb{R}^1\pi_*(\underbrace{\Omega^*_{\mathfrak{X}_{univ}/\mathcal{M}}}_{\text{de Rham complex.}})$

- $\nabla : \underline{H}^1_{DR} \longrightarrow \underline{H}^1_{DR} \otimes_{\mathcal{O}_\mathcal{M}} \Omega^1_{\mathcal{M}/A}$

  GAUSS-MANIN CONNECTION (see [KO68], [HT93, §1.2])

- By using ∇, we have

  $\Omega^1_{\mathcal{M}/A} \simeq \underline{\omega}^{\otimes 2}$ as invertible $\mathcal{O}_F \otimes \mathcal{O}_\mathcal{M}$-sheaves KODAIRA-SPENSOR ISOMORPHISM.

Set $R := A[1/d_F]$ $\left(\text{i.e., } R = \begin{cases} \overline{\mathbb{Q}} & \text{if } p \text{ ramifies in } F/\mathbb{Q} \\ A & \text{otherwise} \end{cases}\right)$

$\boxed{\text{on } R}$ $\underline{\omega} \simeq \bigoplus_{\nu \in I_F} \underline{\omega}(\nu)$ $\quad \underline{\omega}(\nu)$: maximal quotient where $\mathcal{O}_F$ acts via ν. (invertible $\mathcal{O}_{\mathcal{M}}$-sheaf)

$$\rightsquigarrow \underline{\omega}^{\otimes 2} \simeq \bigoplus_{\nu \in I_F} \underline{\omega}(2\nu)$$

■ ∇ induces (on R)

$$\begin{array}{ccc}
\mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(\underline{H}^1_{DR}) & \xrightarrow{\nabla} & \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(\underline{H}^1_{DR}) \otimes_{\mathcal{O}_{\mathcal{M}}} \underline{\omega}^{\otimes 2} \\
 & & \downarrow \text{projection} \\
 & & \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(\underline{H}^1_{DR}) \otimes_{\mathcal{O}_{\mathcal{M}}} \underline{\omega}(2\nu) \\
 & & \downarrow \wr \\
 & & \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(\underline{H}^1_{DR}) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Symm}^2_{\mathcal{O}_{\mathcal{M}}}(\underline{\omega}) \\
 & & \downarrow \text{HODGE FILTRATION } \underline{\omega} \hookrightarrow \underline{H}^1_{DR} \\
 & & \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(\underline{H}^1_{DR}) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Symm}^2_{\mathcal{O}_{\mathcal{M}}}(\underline{H}^1_{DR}) \\
 & & \downarrow \text{product} \\
 & \searrow^{\nabla(\nu)} & \mathrm{Symm}^{k+2}_{\mathcal{O}_{\mathcal{M}}}(\underline{H}^1_{DR})
\end{array}$$

$\nabla(\nu)$ differential operator of degree 2

Recall that for $\mathbb{k} \in \mathbb{Z}[I_F]$,

weight $\mathbb{k}$ τ-HMF $f \in M_{\mathbb{k}}(\tau, \Gamma_{00}(p^\infty); R)$

(i.e., f satisfies (MF1) and (MF2) in Nuccio's talk and satisfies (MF3)$_{\mathbb{k}}$. $\forall \alpha \in (\mathcal{O}_F \otimes S)^\times$, $f((X/S, \lambda, \alpha^{-1}\omega, i)) = \alpha^{\mathbb{k}} f((X/S, \lambda, \omega, i))$)

is regarded as an element of $H^0(\mathcal{M}, \underline{\omega}(\mathbb{k}))$.

where $\underline{\omega}(\mathbb{k}) := \bigotimes_{\nu \in I_F} \underline{\omega}(\nu)^{\otimes k_\nu}$

(refer to NUCCIO'S TALK)

Hence <u>if HODGE FILTRATION $\underline{\omega} \hookrightarrow \underline{H}^1_{DR}$ splits</u>

(i.e., if $^\exists \mathrm{Split} \subseteq \underline{H}^1_{DR}$ s.t., $\underline{H}^1_{DR} = \underline{\omega} \oplus \mathrm{Split}$), we have

the 1-st projection $\underline{H}^1_{DR} \twoheadrightarrow \underline{\omega}$ and

$$\underline{\omega}(\mathbb{k}) \hookrightarrow \mathrm{Symm}^k_{\mathcal{O}_\mathcal{M}}(\underline{\omega}) \hookrightarrow \mathrm{Symm}^k_{\mathcal{O}_\mathcal{M}} \underline{H}^1_{DR} \xrightarrow{\nabla(\nu)} \mathrm{Symm}^{k+2}_{\mathcal{O}_\mathcal{M}} \underline{H}^1_{DR}$$

$$(k = |\mathbb{k}| = \sum_{\nu \in I_F} k_\nu) \qquad \xrightarrow{\text{projection}} \underline{\omega}(\mathbb{k} + 2\nu)$$

$\partial_{\mathbb{k}}(\nu; \mathrm{Split})$

$\rightsquigarrow$ we obtain the differential operator $\partial_{\mathbb{k}}(\nu; \mathrm{Split})$ acts on the space of ε-HMF $M_{\mathbb{k}}(\varepsilon, \Gamma_{00}(p^\infty))$

⊙ Now let us set $x = (X(\mathfrak{A}_j), \lambda(\delta), i(\mathfrak{A}_j))_{/A}$ and

pull back these diagrams by $R \xrightarrow{x_{/R}} \mathcal{M}$ where $x_{/R}$ is identified with the corresponding R-rational point of $\mathcal{M}$.

Denote $\underline{H}^1_{DR}(x)$, $\underline{\omega}(x)$ etc... for pulled-back objects

on $\mathbb{C}$ or $\mathcal{D}_p$, we have $\boxed{\text{canonical splittings}}$ of $\underline{\omega}(x) \hookrightarrow \underline{H}^1_{DR}(x)$

on $\mathbb{C}$: in the category of (compact) differentiable manifolds

$$H^1_{DR}(x)_{/\mathbb{C}} \simeq \underbrace{H^{1,0}(x)}_{\substack{\| \\ \underline{\omega}(x)_{/\mathbb{C}}}} \oplus H^{0,1}(x)$$ ... HODGE DECOMPOSITION

...... Split(HODGE)

on D_p (or $D_p[1/d_F]$)

DWORK - KATZ DECOMPOSITION

$$H^1_{DR}(x) \simeq \underline{\omega}(x)_{/D_p} \oplus \underbrace{U(x)}_{\text{unit root part}}$$ ([Dw71, THEOREM 4.1], [Kat73a])

... Split(DK)

[using the fact that $(X(\mathfrak{A}_j), \lambda(\delta), i(\mathfrak{A}_j))_{/D_p}$ has. good ordinary reduction]

$\boxed{\text{CAUTION}}$ on A or R.

we DO NOT have such "nice" splitting in general.

hence use "CM" (!!)

§ SPLITTINGS OF HODGE FILTRATIONS

KEY LEMMA [Kat78, KEY LEMMA (5.1.27)]

Assume x : CM point. Then

$\exists\ \mathrm{Split}(x)(CM)$ s.t., $H^1_{DR}(x) \simeq \underline{\omega}(x) \oplus \mathrm{Split}(x)(CM)\ /A$

$\mathrm{Split}(x)(CM) \subseteq H^1_{DR}(x)$

and

$$\begin{tikzcd} & H^{0,1}(x) \\ \mathrm{Split}(x)(CM) \arrow[ur, squiggly, "\text{basechange via } \iota_\infty"] \arrow[dr, squiggly, "\text{via } \iota_p"'] & \\ & U(x) \end{tikzcd}$$

[PROOF (ROUGH SKETCH)]

VIA $\quad \mathcal{O}_K \otimes A \xrightarrow{\sim} (\mathcal{O}_F \otimes A) \times (\mathcal{O}_F \otimes A)\ ;\ x \mapsto (\varphi_\Sigma(x), \varphi_\Sigma(\bar{x}))$

we obtain decomposition $\quad \Sigma \overset{\text{VIA}}{\curvearrowright} \mathcal{O}_K \overset{\text{VIA}}{\curvearrowleft} \bar{\Sigma}$

$$H^1_{DR}(x) \simeq \underbrace{\underline{H^1_{DR}(x)(\Sigma)}}_{\substack{\| \text{ "easy"} \\ \underline{\omega}(x)}} \oplus \underbrace{\boxed{H^1_{DR}(x)(\bar{\Sigma})}}_{\substack{\| \\ \mathrm{Split}(x)(CM)}} \cdots \mathrm{Split}(CM)$$

$\blacksquare$

◎ Finally we need to describe the explicit action of

$$\begin{tikzcd} & \vartheta_k(\nu\,;\,\mathrm{Split}(\mathrm{HODGE})) \\ \vartheta_k(\nu\,;\,\mathrm{Split}(CM)) \arrow[ur, squiggly, "\iota_\infty"] \arrow[dr, squiggly, "\iota_p"'] & \\ & \vartheta_k(\nu\,;\,\mathrm{Split}(DK)) \end{tikzcd}$$

In explicit computation of $\vartheta_{\mathbb{k}}(\nu; \mathrm{Split}(\mathrm{HODGE}))$ (resp. $\vartheta_{\mathbb{k}}(\nu; \mathrm{Split}(\mathrm{DK}))$)

we use ω_{trans} (resp. ω_{can})

- $\omega_{\mathrm{trans}} \longleftrightarrow (dz_\nu)_{\nu \in I_F}$ for the canonical coordinates of

$$\mathrm{Lie}(X(\mathfrak{A}_j)/\mathbb{C}) \simeq \mathbb{C}^{I_F} (\simeq \mathbb{C}^{\Sigma})$$

by using these coordinates, the action of $\vartheta_{\mathbb{k}}(\nu; \mathrm{Split}(\mathrm{HODGE}))$

is described as $\dfrac{1}{2\pi\sqrt{-1}}\left(\dfrac{\partial}{\partial z_\nu} + \dfrac{k_\nu}{2\sqrt{-1}\,\mathrm{Im}(z_\nu)}\right)$

… (classical) MAASS-SHIMURA's differential operator.

- $\vartheta_{\mathbb{k}}(\nu; \mathrm{Split}(\mathrm{DK}))$ acts on the q-expansion of an element of $V(\mathfrak{c}; \Gamma_{00}(p^\infty); \mathcal{D}_p)$ via $\theta(\nu)$ defined yesterday.

$$\begin{tikzcd}[column sep=small]
\underset{\vartheta_{\mathbb{k}}(\nu; \mathrm{Split}(\mathrm{HODGE}))}{\overset{\text{“}\frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial}{\partial z_\nu} + \frac{k_\nu}{2\sqrt{-1}\,\mathrm{Im}\, z_\nu}\right)\text{”}}{\parallel}} \ \text{w.r.t. } \omega_{\mathrm{trans}} \quad \mathbb{C} & & \mathcal{D}_p \quad \underset{\vartheta_{\mathbb{k}}(\nu; \mathrm{Split}(\mathrm{DK}))}{\overset{\text{“}\theta(\nu)\text{”}}{\parallel}} \ \text{w.r.t. } \omega_{\mathrm{can}} \\
& (X(\mathfrak{A}_j), \lambda(\sigma), i(\mathfrak{A}_j))/A \arrow[ul, "\iota_\infty"] \arrow[ur, "\iota_p"'] & \\
& \vartheta_{\mathbb{k}}(\nu; \mathrm{Split}(\mathrm{CM})) \arrow[uul, bend left, mapsto] \arrow[uur, bend right, mapsto] &
\end{tikzcd}$$

By using these, we can compute

$$\Omega_p^{-k\Sigma - 2d} \int_{Cl_K(p^\infty)} \hat{\lambda}\, d\mu$$

$$= \Omega_{CM}^{-k\Sigma - 2d} \cdot \cdot \sum_{j=1}^{h} \lambda(\mathfrak{A}_j)^{-1} \cdot \sum_{\substack{\ell \in \mathfrak{A}_j[1/p] \cap \mathcal{P}V_p\mathfrak{A}_j \\ \mathrm{mod}\ \mathcal{O}_F^\times}} \frac{\mathcal{P}\phi'(\ell) \cdot \ell^{\kappa}}{(N(\ell))^{k} |N(\ell)|^{2s}} \Bigg|_{s=0}.$$

$$\phi'(a,b) = \lambda_{\Sigma_p}(a)\, \lambda_{\overline{\Sigma}_p}(b)^{-1}$$

$\mathcal{P}\phi' \rightsquigarrow$ "product of local GAUSS sums"

☞ SUPPLIMENTARY COMMENTS

1°) ON PERIODS. for $f \in M_{\mathbb{k}}(\mathfrak{c}, \Gamma_{00}(p^\infty); A)$
(e.g., $f = G_{k,\phi'}$)

and an element $\omega \in \underline{\omega}(x)$ $\quad x = (X(\mathfrak{A}_j), \lambda(\delta), i(\mathfrak{A}_j))_{/A}$.

$$[\partial_{\mathbb{k}}(d;\mathrm{Split}(CM))f](x)\,\omega(\mathbb{k}+2d) \in \underline{\omega}(x)(\mathbb{k}+2d) \quad \Big(\omega = \sum_{\nu \in I_F} \omega(\nu),\ \ \omega(\mathbb{k}+2d) = \prod_{\nu \in I_F} \omega(\nu)^{k_\nu + 2d_\nu}\Big).$$

$$[\partial_{\mathbb{k}}(d;\mathrm{Split}(CM))f](x)\,\omega(\mathbb{k}+2d) \xrightarrow{\ \iota_\infty\ } [\partial_{\mathbb{k}}(d;\mathrm{Split}(\mathrm{HODGE}))f](x)\,\omega(\mathbb{k}+2d) = [\partial_{\mathbb{k}}(d;\mathrm{Split}(\mathrm{HODGE}))f](x)\,\Omega_{CM}^{\mathbb{k}+2d}\,\omega_{trans}(\mathbb{k}+2d)$$

$$[\partial_{\mathbb{k}}(d;\mathrm{Split}(CM))f](x)\,\omega(\mathbb{k}+2d) \xrightarrow{\ \iota_p\ } [\partial_{\mathbb{k}}(d;\mathrm{Split}(DK))f](x)\cdot\omega(\mathbb{k}+2d) = [\partial_{\mathbb{k}}(d;\mathrm{Split}(DK))f](x)\,\Omega_p^{\mathbb{k}+2d}\,\omega_{can}(\mathbb{k}+2d)$$

$$\left[\text{by definition,}\quad \begin{aligned} \omega &= \Omega_{CM}\,\omega_{trans} && (\text{on } \mathbb{C}) \\ &= \Omega_p\,\omega_{can} && (\text{on } D_p) \end{aligned}\right]$$

Since $\frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial}{\partial z_\nu} + \frac{k_\nu}{2\sqrt{-1}\,\mathrm{Im}\,z_\nu}\right)$ (resp. $\theta(\nu)$) is the description

of $\partial_{\mathbb{k}}(\nu;\mathrm{Split}(\mathrm{HODGE}))$ (resp. $\partial_{\mathbb{k}}(\nu;\mathrm{Split}(DK))$)

w.r.t. ω_{trans} (resp. w.r.t. ω_{can}), we need periods Ω_{CM}, Ω_p

to compare explicit calculations of differential operators between archimedean side and p-adic side.

2°) ON "INTEGRALITY OF $\vartheta(\nu ; \mathrm{Split}(\mathcal{D}K))$"

By naive definition,

$$\vartheta(\nu ; \mathrm{Split}(\mathcal{D}K)) f \in V(\tau, \Gamma_{00}(p^\infty) ; \mathcal{D}_p) \otimes \mathcal{D}_p[1/d_F]$$

for $f \in V(\tau, \Gamma_{00}(p^\infty) ; \mathcal{D}_p)$.

(Since we use the decomposition $\underline{\omega} \simeq \bigoplus_{\nu \in I_F} \underline{\omega}(\nu)$ on $\mathcal{D}_p[1/d_F]$)

HOWEVER, since $\vartheta(\nu ; \mathrm{Splits}(\mathcal{D}K))$ acts on the q-expansion of $f \in \mathcal{D}_p[[q(\mathfrak{a}\mathfrak{b})]]$ via.

$$\theta(\nu) f = \theta(\nu)\left(\sum a_\alpha q^\alpha\right) = \sum \nu(\alpha) a_\alpha q^\alpha \in \mathcal{D}_p[[q(\mathfrak{a}\mathfrak{b})]]$$

Hence by q-expansion principle, $\vartheta(\nu ; \mathrm{Split}(\mathcal{D}K)) f \in V(\tau ; \Gamma_{00}(p^\infty) ; \mathcal{D}_p)$

A POSTEORI,

and $\vartheta(\nu ; \mathrm{Split}(\mathcal{D}K))$ preserves "integrality"