

DAY 3 5th Apr 2012

No. DAY 3 - 1/10

CONSTRUCTION OF p -ADIC L-FUNCTIONS FOR CM-FIELDS À LA KATZ, HIDA, TILLOINE (CONTINUED)

- REVIEW
- ARITHMETIC DIFFERENTIAL OPERATORS
- SPLITTINGS OF HODGE FILTRATIONS
- "BULLET TOUR" AROUND MAIN CONJECTURES
→ Sorry, but "TOUR CANCELLED"

NOTATION

p : prime number. (K, Σ) : p -ordinary CM-type
 U
 F : maximal totally real subfield

$\tau \subseteq F$: fractional ideal prime to p (polarization ideal)

REVIEW.

Yesterday, we constructed the EISENSTEIN MEASURE.

$$\mathbb{E}_\tau: \text{Cont}(G, R) \longrightarrow V(\tau, \Gamma_\infty(p^\infty); R)$$

$$\begin{array}{ccc} \phi & \longrightarrow & G_{1, \phi'}^{(p)} \end{array}$$

$\downarrow$ closure.
 $G := (\mathcal{O}_F \otimes \mathbb{Z}_p)^{x, 2} / \mathcal{O}_F^{x, 2}$
 $R = \varprojlim_n R/p^n R$

From class field theory.

$$G \longrightarrow \text{Cl}_K(p^\infty) \longrightarrow \text{Cl}_K \longrightarrow 0$$

$\{2l_j^{-1}\}_{j=1}^h$ representative of Cl_K
 $2l_j \subseteq \mathcal{O}_K$, prime to p .

$$\rightsquigarrow \mu : \text{Cont}(\text{Cl}_K(p^\infty), \mathbb{D}_p) \longrightarrow \mathbb{D}_p.$$

$$\overset{\psi}{f} \longmapsto \sum_{j=1}^h \int_G \overset{\psi}{f}([2]_j^{-1} g) dE([2]_j)$$

where

$$E([2]_j) := E_{\tau N_{K/F}([2]_j)^{-1}}((X([2]_j), \lambda(\delta), i([2]_j)) / \mathbb{D}_p)$$

Let λ : Größencharacter of type (A_0) of infinite type κ

$$\kappa = k \sum_{\nu \in \Sigma} \sum_{+} d\nu(\sigma - \bar{\sigma}) \quad d\nu := (d\nu)_{\nu \in \Sigma}$$

By definition.

$$\int_{\text{Cl}_K(p^\infty)} \hat{\lambda} d\mu = \sum_{j=1}^h \lambda([2]_j)^{-1} \underbrace{G_{\tau, 1, N(a)^{k-1} \lambda_{\overline{\tau}}(a) \lambda_{\overline{\tau}}(b)^{-1}}^{(p)}}_{\substack{\text{yesterday's observation} \\ + q\text{-expansion principle}}} (X([2]_j), \lambda(\delta), i([2]_j)) / \mathbb{D}_p$$

$$\theta(d\nu) G_{\tau, 1, \lambda_{\overline{\tau}}(a) \lambda_{\overline{\tau}}(b)^{-1}}^{(p)}$$

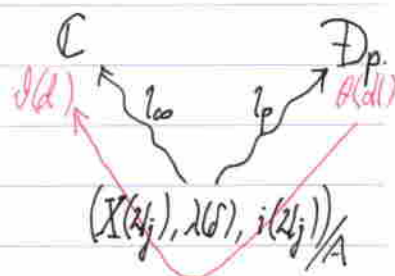
has analytic description.

where for $\nu \in I_F (\simeq \Sigma)$

$$\theta(\nu) \left(\sum_{\alpha \in \alpha b} a_\alpha g^\alpha \right) = \sum_{\alpha \in \alpha b} \nu(\alpha) a_\alpha g^\alpha \quad \text{differential operator}$$

KATZ' "L'EXÉCUTION TRANSCENDANTE"

If $\theta(d)$ comes from a differential operator $/A$,
we can compute "evaluation at CM-points"
by using "archimedean description,"
(i.e., DIRICHLET SERIES expression)



ARITHMETIC DIFFERENTIAL OPERATORS. ([KAT 78, CHAPTER II])

$\mathcal{M} := \mathcal{M}(\pi, \Gamma_{\infty}(p^{\infty}))/A$ moduli of π -HBAV of level $\Gamma_{\infty}(p^{\infty})$.

$(\pi: \mathcal{X}_{\text{univ}} \rightarrow \mathcal{M}, \lambda_{\text{univ}}, i_{\text{univ}})$: universal π -HBAV of level $\Gamma_{\infty}(p^{\infty})$

Set $\underline{\omega} := \pi_* \Omega^1_{\mathcal{X}_{\text{univ}}/\mathcal{M}}$, $\underline{H}^1_{\text{DR}} := R^1_{\pi_*}(\Omega^*_{\mathcal{X}_{\text{univ}}/\mathcal{M}})$
 $\uparrow$ de Rham complex.

• $\nabla: \underline{H}^1_{\text{DR}} \rightarrow \underline{H}^1_{\text{DR}} \otimes_{\mathcal{O}_{\mathcal{M}}} \Omega^1_{\mathcal{M}/A}$

GAUSS-MANIN CONNECTION (see [KOG8], [HT93, §1.2])

• By using ∇ , we have

$\Omega^1_{\mathcal{M}/A} \simeq \underline{\omega}^{\otimes 2}$

as invertible $\mathcal{O}_{\mathcal{M}} \otimes \mathcal{O}_{\mathcal{M}}$ -sheaves

KODAIRA-SPENCER ISOMORPHISM.

Set $R := A[1/d_F]$ (i.e., $R = \begin{cases} \mathbb{Q} & \text{if } p \text{ ramifies in } F/\mathbb{Q} \\ A & \text{otherwise} \end{cases}$)

on R $\underline{\omega} \simeq \bigoplus_{\nu \in I_F} \underline{\omega}(\nu)$ $\underline{\omega}(\nu)$: maximal quotient where $\mathcal{O}_F$ acts via ν (invertible $\mathcal{O}_M$ -sheaf)

$$\rightsquigarrow \underline{\omega}^{\otimes 2} \simeq \bigoplus_{\nu \in I_F} \underline{\omega}(2\nu)$$

∇ induces (on R)

$$\begin{array}{c} \text{Sym}_{\mathcal{O}_M}^k(H_{\mathbb{D}R}^1) \xrightarrow{\nabla} \text{Sym}_{\mathcal{O}_M}^k(H_{\mathbb{D}R}^1) \otimes_{\mathcal{O}_M} \underline{\omega}^{\otimes 2} \\ \downarrow \text{projection} \\ \text{Sym}_{\mathcal{O}_M}^k(H_{\mathbb{D}R}^1) \otimes_{\mathcal{O}_M} \underline{\omega}(2\nu) \\ \downarrow \\ \text{Sym}_{\mathcal{O}_M}^k(H_{\mathbb{D}R}^1) \otimes_{\mathcal{O}_M} \text{Sym}_{\mathcal{O}_M}^2(\underline{\omega}) \\ \downarrow \text{HODGE FILTRATION } \underline{\omega} \hookrightarrow H_{\mathbb{D}R}^1 \\ \text{Sym}_{\mathcal{O}_M}^k(H_{\mathbb{D}R}^1) \otimes_{\mathcal{O}_M} \text{Sym}_{\mathcal{O}_M}^2(H_{\mathbb{D}R}^1) \\ \downarrow \text{product} \\ \text{Sym}_{\mathcal{O}_M}^{k+2}(H_{\mathbb{D}R}^1) \end{array}$$

$\nabla(\nu)$
differential operator
of degree 2

Recall that for $k \in \mathbb{Z}[I_F]$,

weight k π -HMF $f \in M_k(\pi, \Gamma_{\infty}(p^{\infty}); R)$

(i.e., f satisfies (MF1) and (MF2) in Nuccio's talk and satisfies (MF3) $_k$. $\forall \alpha \in (\mathcal{O}_F \otimes S)^{\times}$, $f((X/S, \lambda, \alpha^{-1}\omega, i)) = \alpha^k f((X/S, \lambda, \omega, i))$)

is regarded as an element of $H^0(M, \underline{\omega}(k))$.

$$\text{where } \underline{\omega}(k) := \bigotimes_{\nu \in I_F} \underline{\omega}(\nu)^{\otimes k_\nu}$$

(refer to NUCCIO'S TALK)

Hence if HODGE FILTRATION $\underline{\omega} \hookrightarrow H_{\text{DR}}^1$ splits

(i.e., if $\exists \text{ Split} \subseteq H_{\text{DR}}^1$ s.t., $H_{\text{DR}}^1 = \underline{\omega} \oplus \text{Split}$), we have

the 1-st projection $H_{\text{DR}}^1 \twoheadrightarrow \underline{\omega}$ and

$$\begin{aligned} \underline{\omega}(k) \hookrightarrow \text{Sym}_{\mathcal{O}_M}^k(\underline{\omega}) \hookrightarrow \text{Sym}_{\mathcal{O}_M}^k H_{\text{DR}}^1 &\xrightarrow{\nabla(\nu)} \text{Sym}_{\mathcal{O}_M}^{k+2} H_{\text{DR}}^1 \\ &\xrightarrow{\text{projection}} \underline{\omega}(k+2\nu) \end{aligned}$$

$\rightsquigarrow$ we obtain the differential operator $d_k(\nu; \text{Split})$
acts on the space of τ -HMF $M_k(\tau, \Gamma_\infty(p^\infty))$

⊙ Now let us set $x = (X(2j), \lambda(\delta), i(2j))_A$ and

pull back these diagrams by $R \xrightarrow{x_R} M$ where x_R is identified
with the corresponding R -rational point of M .

Denote $H_{\text{DR}}^1(x)$, $\underline{\omega}(x)$ etc... for pulled-back objects.

on $\mathbb{C}$ or $\mathbb{D}_p$, we have canonical splittings of $\underline{\omega}(x) \hookrightarrow H_{\text{DR}}^1(x)$

on $\mathbb{C}$: in the category of (compact) differentiable manifolds

$$H_{\mathcal{D}R}^1(x)/\mathbb{C} \simeq \underbrace{H^{1,0}(x)}_{\omega(x)/\mathbb{C}} \oplus H^{0,1}(x) \quad \dots \text{HODGE DECOMPOSITION}$$

..... Split (HODGE)

on $\mathbb{D}_p$ (or $\mathbb{D}_p[1/d_F]$)

$$H_{\mathcal{D}R}^1(x) \simeq \omega(x)/\mathbb{D}_p \oplus \underbrace{U(x)}_{\text{unit root part}} \quad (\text{DWORK-KATZ DECOMPOSITION})$$

([Dw71, THEOREM 4.1], [Kat73a])

..... Split ($\mathcal{D}R$)

using the fact that $(X(2l_j), \lambda(\sigma), i(2l_j))/\mathbb{D}_p$ has
good ordinary reduction

CAUTION on A or $\mathbb{R}$.

we DO NOT have such "nice" splitting in general.

hence use "CM" (!!)

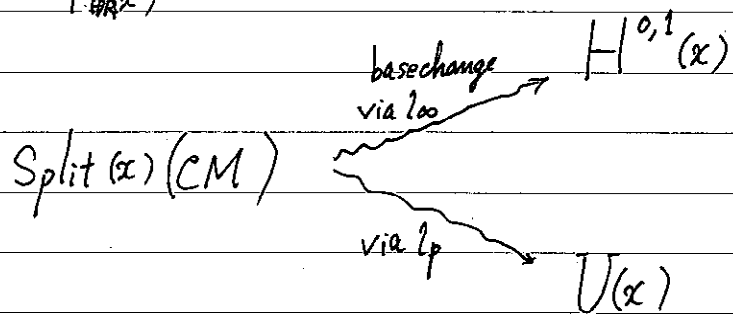
§ SPLITTINGS OF HODGE FILTRATIONS

KEY LEMMA [Kat78, KEY LEMMA (5.1.27)]

Assume x : CM point. Then

$$\exists \text{ Split}(x)(CM) \text{ s.t., } H_{\mathcal{D}R}^1(x) \simeq \omega(x) \oplus \text{Split}(x)(CM) / A$$

and



[PROOF (ROUGH SKETCH)]

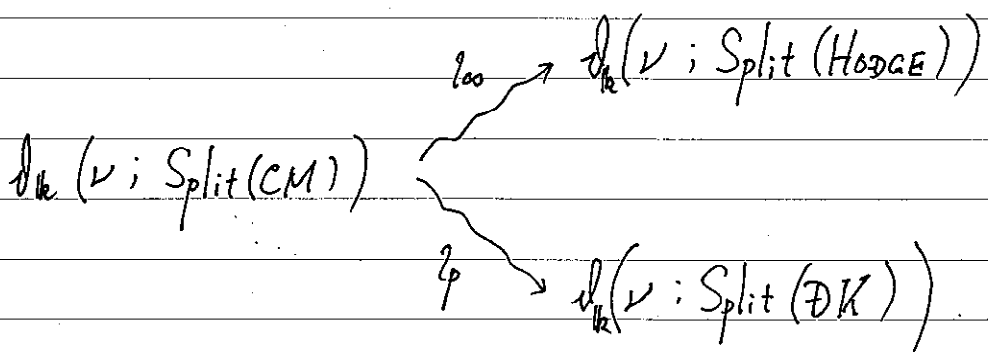
(VIA) $\mathcal{O}_K \otimes A \xrightarrow{\sim} (\mathcal{O}_F \otimes A) \times (\mathcal{O}_F \otimes A) ; x \mapsto (\varphi_{\Sigma}(x), \varphi_{\bar{\Sigma}}(x))$

we obtain decomposition

$$H_{\mathcal{D}R}^1(x) \simeq \underbrace{H_{\mathcal{D}R}^1(x)(\Sigma)}_{\parallel \text{"easy"} \parallel \omega(x)} \oplus \underbrace{H_{\mathcal{D}R}^1(x)(\bar{\Sigma})}_{\parallel \text{Split}(x)(CM) \parallel} \dots \text{Split}(CM)$$

Diagram showing the decomposition of $H_{\mathcal{D}R}^1(x)$. The first term $H_{\mathcal{D}R}^1(x)(\Sigma)$ is labeled "easy" and $\omega(x)$. The second term $H_{\mathcal{D}R}^1(x)(\bar{\Sigma})$ is labeled "Split(x)(CM)". Arrows labeled "VIA Σ " and "VIA $\bar{\Sigma}$ " point from $\mathcal{O}_K$ to the respective terms.

Finally we need to describe the explicit action of



In explicit computation of $d_k(\nu; \text{Split}(\text{HODGE}))$ (resp. $d_k(\nu; \text{Split}(\mathbb{D}K))$)
we use ω_{trans} (resp. ω_{can})

• $\omega_{\text{trans}} \longleftrightarrow (dz_v)_{v \in I_F}$ for the canonical coordinates of
 $\text{Lie}(X(2)_j)/\mathbb{C} \simeq \mathbb{C}^{I_F} \simeq \mathbb{C}^{\Sigma}$

by using these coordinates, the action of $d_k(\nu; \text{Split}(\text{HODGE}))$

is described as $\frac{1}{2\pi\sqrt{-1}} \left(\frac{\partial}{\partial z_v} + \frac{k_v}{2\sqrt{-1} \text{Im}(z_v)} \right)$

... (classical) MAASS-SHIMURA's differential operator.

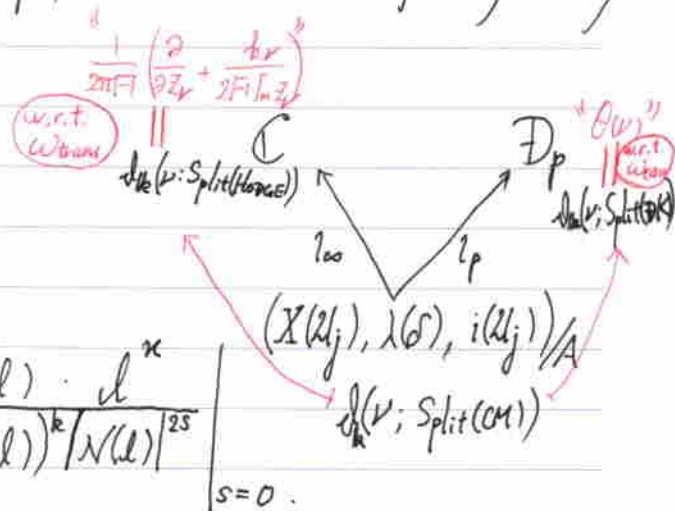
• $d_k(\nu; \text{Split}(\mathbb{D}K))$ acts on the q -expansion

of an element of $V(\kappa, \Gamma_0(p^\infty); \mathbb{D}_p)$ via $\theta(\nu)$ defined yesterday.

By using these, we can compute

$$\Omega_p^{-k\Sigma-2d} \int_{\text{cl}_K(p^\infty)} \hat{\lambda} d\mu$$

$$= \Omega_{\text{CM}}^{-k\Sigma-2d} \cdot \left(\bigoplus \right) \cdot \sum_{j=1}^h \lambda(2)_j^{-1} \sum_{\substack{l \in 2_j \Gamma / p \Gamma \cap P_V 2_j \\ \text{mod } \mathcal{O}_F^\times}} \frac{P\phi'(l) \cdot l^\kappa}{(N(l))^k |N(l)|^{2s}} \Big|_{s=0}$$



$$\phi'(a, b) = \lambda_{\Sigma_p}(a) \lambda_{\Sigma_p}(b)^{-1}$$

$P\phi' \rightsquigarrow$ "product of local GAUSS sums"



☞ SUPPLEMENTARY COMMENTS

1°) ON PERIODS.

for $f \in M_k(\Gamma, \Gamma_\infty(p^\infty); A)$
(e.g., $f = G_k, \phi'$)

and an element $\omega \in \omega(x)$ $x = (X(2j), \lambda(\delta), i(2j)) / A$.

$$[d_k(d; \text{Split}(CM))f](x) \omega(k+2d) \in \omega(x)(k+2d) \quad \left(\omega = \sum_{\nu \in I_F} \omega(\nu), \quad \omega(k+2d) = \prod_{\nu \in I_F} \omega(\nu)^{k_\nu+2d_\nu} \right)$$

$$[d_k(d; \text{Split}(CM))f](x) \omega(k+2d) \xrightarrow{I_\infty} [d_k(d; \text{Split}(Hodge))f](x) \omega(k+2d) \parallel [d_k(d; \text{Split}(Hodge))f](x) \Omega_{CM}^{k+2d} \omega_{trans}(k+2d)$$

$$\xrightarrow{I_p} [d_k(d; \text{Split}(\mathbb{D}K))f](x) \cdot \omega(k+2d)$$

$$\parallel [d_k(d; \text{Split}(\mathbb{D}K))f](x) \Omega_p^{k+2d} \omega_{can}(k+2d)$$

$$\left\{ \begin{array}{l} \text{by definition, } \omega = \Omega_{CM} \omega_{trans} \quad (\text{on } \mathbb{C}) \\ \quad \quad \quad = \Omega_p \omega_{can} \quad (\text{on } \mathbb{D}_p) \end{array} \right\}$$

Since $\frac{1}{2\pi i - 1} \left(\frac{2}{\partial Z_\nu} + \frac{k_\nu}{2F-1 \text{Im} Z_\nu} \right)$ (resp. $\theta(\nu)$) is the description

of $d_k(\nu; \text{Split}(Hodge))$ (resp. $d_k(\nu; \text{Split}(\mathbb{D}K))$)

w.r.t. ω_{trans} (resp. w.r.t. ω_{can}), we need periods Ω_{CM}, Ω_p

to compare explicit calculations of differential operators
between archimedean side and p-adic side.

2°) ON "INTEGRALITY OF $\mathcal{I}(\nu; \text{Split}(\mathbb{D}K))$ "

By naive definition,

$$\mathcal{I}(\nu; \text{Split}(\mathbb{D}K))f \in V(\pi, \Gamma_{\infty}(p^{\infty}); \mathbb{D}_p) \otimes \mathbb{D}_p[[1/d_F]]$$

for $f \in V(\pi, \Gamma_{\infty}(p^{\infty}); \mathbb{D}_p)$.

(Since we use the decomposition $\omega \simeq \bigoplus_{\nu \in \Gamma_F} \omega(\nu)$ on $\mathbb{D}_p[[1/d_F]]$)

HOWEVER, since $\mathcal{I}(\nu; \text{Split}(\mathbb{D}K))$ acts on the q -expansion

of $f \in \mathbb{D}_p[[q(\alpha/b)]]$ via

$$\mathcal{I}(\nu)f = \mathcal{I}(\nu)\left(\sum a_x q^x\right) = \sum \nu(x) a_x q^x \in \mathbb{D}_p[[q(\alpha/b)]]$$

Hence by q -expansion principle, $\mathcal{I}(\nu; \text{Split}(\mathbb{D}K))f \in V(\pi, \Gamma_{\infty}(p^{\infty}); \mathbb{D}_p)$

A POSTERIORI,

and $\mathcal{I}(\nu; \text{Split}(\mathbb{D}K))$ preserves "integrality"

