No.

INTERPOLATION FORMULA FOR KATZ-HIDA-TILOUIN MEASURE

p : prime number $\quad (K, \Sigma)$: p-ordinary CM type $\quad$ fix $\overline{\mathbb{Q}} \xrightarrow{\iota_\infty} \mathbb{C}$, $\overline{\mathbb{Q}} \xrightarrow{\iota_p} \overline{\mathbb{Q}}_p$

F : maximal totally real subfield

$\mathfrak{D}_p$: valuation ring of $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$

$\mathfrak{C} \subseteq O_K$ integral ideal prime to p.

decompose $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ s.t., $\quad \left(\begin{smallmatrix}\overline{\mathfrak{F}} \\ \| \end{smallmatrix}\right)$

- $\mathfrak{F}\mathfrak{F}_c$ consists of "splitting prime", $\mathfrak{F} + \mathfrak{F}_c = O_K$, $\mathfrak{F} + \mathfrak{F}^c = O_K$, $\mathfrak{F}_c + \mathfrak{F}_c^c = O_K$, $\mathfrak{F}_c \supseteq \mathfrak{F}^c$
- $\mathfrak{I}$ consists of ideals inert or ramified in K/F

Choose $\delta \in K$ totally imaginary satisfying

$$(*)\begin{cases} \cdot \text{ image of } \delta \text{ in } K\otimes\mathbb{R} \xrightarrow{\sim} F\otimes\mathbb{C} \text{ is contained in } F\otimes\mathbb{R} \text{ and totally positive.} \\ \cdot\ \langle u, v\rangle := (\bar{u}v - u\bar{v})/2\delta \text{ induces an isomorphism} \\ \quad O_K \wedge O_K \xrightarrow{\sim} \mathfrak{d}_F^{-1}\mathfrak{c}^{-1} \text{ for a fractional ideal } \mathfrak{c} \subseteq F \text{ prime to } p\mathfrak{C}\mathfrak{C}^c \end{cases}$$

LOCAL GAUSS SUM AT PRIME $\mathfrak{Q} \subseteq O_K$

$\lambda_{\mathfrak{Q}} : K_{\mathfrak{Q}}^\times \longrightarrow \overline{\mathbb{Q}}^\times$ conductor $\mathfrak{Q}^{e(\mathfrak{Q})}$

$$G(-2\delta_{\mathfrak{Q}}; \lambda_{\mathfrak{Q}}) \overset{def}{:=} \lambda_{\mathfrak{Q}}(\varpi_{\mathfrak{Q}})^{-e(\mathfrak{Q})} \sum_{u \in (O_{K,\mathfrak{Q}}/\mathfrak{Q}^{e(\mathfrak{Q})})^\times} \lambda_{\mathfrak{Q}}(u)\, e_{\mathfrak{Q}}\left(\varpi_{\mathfrak{Q}}^{-e(\mathfrak{Q})}(-2\delta_{\mathfrak{Q}})^{-1}u\right)$$

where $\varpi_{\mathfrak{Q}} \in O_{K,\mathfrak{Q}}$ uniformizer $\quad e_{\mathfrak{Q}}$: "standard additive character"

(please correct ~~~ part)

No.

THEOREM (HIDA-TILOUINE, [HT93, THEOREM II])

$\exists !$ μ : $\mathcal{D}_p$-valued measure on the ray class group $Cl_K(\mathfrak{C}p^\infty)$ modulo $\mathfrak{C}p^\infty$ of K (depending on the choice of δ satisfying $(*)$)

s.t.
$$\frac{\int_{Cl_K(\mathfrak{C}p^\infty)} \hat{\lambda}\, d\mu}{\Omega_p^{k\Sigma+2d}} = [\mathcal{O}_K^\times : \mathcal{O}_F^\times]\, W_p(\lambda) \frac{(-1)^{k[F:\mathbb{Q}]} \pi^{|d|} \Gamma_\Sigma(k\Sigma+d)}{\sqrt{d_F}\, Im(\delta)^{d}\, \Omega_{CM}^{k\Sigma+2d}}$$
$$\times \prod_{\mathfrak{L}|\mathfrak{C}} (1-\lambda(\mathfrak{L})) \prod_{\mathfrak{P}\in\Sigma_p} (1-\lambda(\mathfrak{P}^c))(1-\check{\lambda}(\mathfrak{P}^c))\, L(0,\lambda)$$

for all größencharacters λ of type (A_0) with conductor dividing $\mathfrak{C}p^\infty$

satisfying (i) any prime factor $\mathfrak{L}$ of $\mathfrak{F}$ divides $cond(\lambda)$

(ii) infinite type $\kappa = k\Sigma + \sum_{\sigma\in\Sigma} d_\sigma(\sigma-\bar{\sigma})$

with $\cdot\ k \geq 1,\ d_\sigma \geq 0\ \ \forall \sigma\in\Sigma$

or $\cdot\ k \leq 1,\ d_\sigma \geq 1-m_\circ\ \ \forall\sigma\in\Sigma$

where d_F : discriminant of F $\quad \check{\lambda} : A_K^\times/K^\times \to \mathbb{C}^\times \quad \check{\lambda}(x) := \lambda(c(x))^{-1}|x|_{A_K}$

$W_p(\lambda) := \prod_{\mathfrak{P}\in\Sigma_p} N\mathfrak{P}^{-e(\mathfrak{P})} G(-2\delta_{\mathfrak{P}} ; \lambda_{\mathfrak{P}})$

(We'll deal with the case $\mathfrak{C}=(1)$ for simplicity i.e. KATZ' CASE)

[CONVENTION ON MULTI-INDEX]

$d := \sum_{\sigma\in\Sigma} d_\sigma \sigma$ (i.e., $\kappa = k\Sigma + d(1-c)$)

$\Omega_*^{k\Sigma+2d} = \prod_{\sigma\in\Sigma} \Omega_*(\sigma)^{k_\sigma+2d_\sigma}$ $\quad * = p$ or CM

$\pi^{|d|} = \pi^{\sum_{\sigma\in\Sigma} d_\sigma}$ $\quad Im(\delta)^{d} = \prod_{\sigma\in\Sigma} Im(\sigma(\delta))^{d_\sigma}$ $\quad \Gamma_\Sigma(k\Sigma+d) = \prod_{\sigma\in\Sigma}\Gamma(k+d_\sigma)$

No.

Hecke's Eisenstein series

$k \geq 1$, $\phi : (\mathcal{O}_F \otimes \mathbb{Z}_p) \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \longrightarrow \mathbb{C}$ locally constant

s.t., $\cdot\ \mathrm{Supp}\, \phi \subseteq (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$

$\cdot\ \phi(e^{-1}x, ey) = N_{F/\mathbb{Q}}(e)^k \phi(x, y) \quad \forall e \in \mathcal{O}_F^\times$

$(\mathcal{L}, \langle\, , \rangle, i)$: $\mathfrak{c}$-polarized lattice in $\mathbb{C}^{I_F}$
($\rightsquigarrow$ $\mathfrak{c}$-polarized HBAV $(X(\mathcal{L}), \lambda(\mathcal{L}), \omega_{\mathrm{trans}}, i(\mathcal{L}))$)

$$G_{k,\phi}((X(\mathcal{L}), \lambda(\mathcal{L}), \omega_{\mathrm{trans}}, i(\mathcal{L})) ; s) := \frac{(-1)^{k[F:\mathbb{Q}]} \Gamma(k+s)^{[F:\mathbb{Q}]}}{\sqrt{d_F}} \sum_{\substack{l \in \mathcal{L}[1/p] \cap PV_p(\mathcal{L}) \\ \text{modulo } E = \mathcal{O}_F^\times}} \frac{P\phi(l)}{N_{F/\mathbb{Q}}(l)^k \left| N_{F/\mathbb{Q}}(l) \right|^{2s}}$$

has analytic continuation on whole $\mathbb{C}$ w.r.t. s

● evaluation at ($\mathfrak{c}$-polarized) Tate variety $(\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1})$

$\mathfrak{a}$: prime to p

$$\left(\mathrm{Tate}(\mathfrak{a}\mathfrak{b}) / \mathbb{C}((q(\mathfrak{a}\mathfrak{b}))), \lambda_{\mathrm{Tate}}, \underset{\omega_{\mathrm{can}}}{\underset{\|}{\omega_{\mathrm{Tate}}}}, i_{\mathrm{Tate}} \right)$$

is analytically "equivalent" to evaluate at Tate lattice $(\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau), \langle\, , \rangle_{\mathrm{can}}, i_{\mathrm{can}})$

defined by $\cdot\ \mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau) = 2\pi\sqrt{-1}\,(\mathfrak{d}_F^{-1}\mathfrak{a} + \mathfrak{b}\tau)$ ($\tau \in K \otimes \mathbb{C}$ with $\mathrm{Im}(\tau)$: totally positive)

$$\boxed{q^\alpha \longleftrightarrow \exp(2\pi\sqrt{-1}\,\mathrm{Tr}(\alpha\tau))}$$

$\cdot\ \langle u, v \rangle_{\mathrm{can}} = \langle 2\pi\sqrt{-1}(a_u + b_u\tau), 2\pi\sqrt{-1}(a_v + b_v\tau) \rangle_{\mathrm{can}} := a_u b_v - a_v b_u$

$\boxed{\mathfrak{a} : \text{prime to } p.}$

$\cdot$ in $K \otimes \mathbb{Z}_p$, $\mathcal{O}_F \otimes \mathbb{Z}_p \equiv \mathfrak{a}^{-1} \otimes \mathbb{Z}_p$

$$i_{\mathrm{can}} : \mathfrak{d}_F^{-1} \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathfrak{d}_F^{-1}\mathfrak{a}^{-1} \otimes \mathbb{Z}_p \xhookrightarrow{\times 2\pi\sqrt{-1}} \mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau) \otimes \mathbb{Z}_p$$

ϕ : constant on cosets modulo p^n $\longrightarrow$ $\mathrm{Supp}\,\mathcal{P}\phi \subseteq \left(\mathfrak{d}_F^{-1}\otimes\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}\right)\times\left(\mathcal{O}\otimes\mathbb{Z}_p\right)^\times$

$$\therefore\ G_{k,\phi}\left(\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau),\ \langle\ ,\ \rangle_{can},\ i_{can},\ s\right)$$
$$= \frac{(-1)^{k[F:\mathbb{Q}]}\,\Gamma(k+s)^{[F:\mathbb{Q}]}}{\sqrt{d_F}\,(2\pi\sqrt{-1})^{k[F:\mathbb{Q}]}\,(2\pi)^{2[F:\mathbb{Q}]s}} \sum_{\substack{b\in\mathfrak{b}\setminus\{0\}\\ \text{modulo } \mathcal{O}_F^\times}} \sum_{a_0\in\mathfrak{d}_F^{-1}\mathfrak{a}^{-1}/p^n} \mathcal{P}\phi\left(\frac{a_0}{p^n},b\right) \underbrace{\boxed{\sum_{a\in\mathfrak{d}_F^{-1}\mathfrak{a}^{-1}} \frac{1}{N\left(a+\frac{a_0}{p^n}+b\tau\right)^k \left|N\left(a+\frac{a_0}{p^n}+b\tau\right)\right|^{2s}}}}_{=\ S_k\left(\frac{a_0}{p^n}+b\tau,\ \mathfrak{d}_F^{-1}\mathfrak{a}^{-1},\ s\right)}$$

$\boxed{\text{KEY INGREDIENT}}$

1° POISSON SUMMATION FORMULA to $t \longmapsto \dfrac{1}{N\left(t+\frac{a_0}{p^n}+b\tau\right)^k \left|N\left(t+\frac{a_0}{p^n}+b\tau\right)\right|^{2s}}$

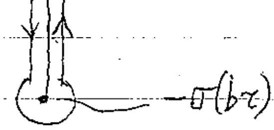
$$\rightsquigarrow S_k\left(\frac{a_0}{p^n}+b\tau,\ \mathfrak{d}_F^{-1}\mathfrak{a}^{-1},\ s\right) = N\mathfrak{a}\sqrt{d_F}\sum_{\lambda\in\mathfrak{a}} C_k\left(\lambda,\ \frac{a_0}{p^n}+b\tau,\ s\right)$$

$$C_k\left(\lambda,\ \frac{a_0}{p^n}+b\tau,\ s\right) = \int_{F\otimes\mathbb{R}} \frac{\exp\left(-2\pi\sqrt{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(\lambda t)\right)}{N\left(t+\frac{a_0}{p^n}+b\tau\right)^k \left|N\left(a+\frac{a_0}{p^n}+b\tau\right)\right|^{2s}}\,dt$$
$$= \exp\left(2\pi\sqrt{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(\lambda a_0/p^n)\right)\underbrace{C_k(\lambda,\ b\tau,\ s)}_{\text{can compute "}\sigma\text{-componentwise"}}$$

(2° PARTIAL FOURIER INVERSION FORMULA:
$$\sum_{a_0\in\mathfrak{d}_F^{-1}\mathfrak{a}^{-1}/p^n} \mathcal{P}\phi\left(\frac{a_0}{p^n},\ b\right)\exp\left(2\pi\sqrt{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(\lambda a_0/p^n)\right) = \phi(\lambda,\ b)$$
)

3° ANALYTIC CONTINUATION OF $C_k(\lambda,\ b\tau,\ s)$: "MULTIPLE CONTOUR INTEGRAL"

$\boxed{\sigma\text{-component}}$



4° RESIDUE CALCULUS

$$\rightsquigarrow C_k(\lambda,\ b\tau,\ 0) = \begin{cases} 0 & \text{unless } \lambda b \gg 0 \\ \dfrac{(2\pi\sqrt{-1})^{k[F:\mathbb{Q}]}\,(-1)^{k[F:\mathbb{Q}]}}{\Gamma(k)^{[F:\mathbb{Q}]}}\,\mathrm{sgn}\left(N_{F/\mathbb{Q}}(\lambda)\right)N_{F/\mathbb{Q}}(\lambda)^{k-1}\,q^{\lambda b} & \lambda b \gg 0 \end{cases}$$

$\left(q^{\alpha} = \exp\left(2\pi\sqrt{-1}\,\mathrm{Tr}_{F/\mathbb{Q}}(\alpha\tau)\right)\right)$