Archimedean zeta integrals for $GL(3,\mathbf{R})\times GL(2,\mathbf{R})$

Taku Ishii (Seikei University)

Reduction to radial parts

- $G\equiv G_n=GL(n,\mathbf{R})$, $K\equiv K_n=O(n)$: maximal compact subgroup of G
- $N\equiv N_n=\{(x_{ij})\in G_n\mid x_{ii}=1, x_{ij}=0\ (i>j)\}$: maximal unipotent subgroup of G
- $G=NAK$: Iwasawa decomposition with $A\equiv A_n=\{y=\mathrm{diag}(y_1y_2\cdots y_n, y_2\cdots y_n,\dots,y_n)\mid y_i>0\}$
- $\psi_{\mathbf{R}}(t)=\exp(2\pi\sqrt{-1}t)$ $(t\in\mathbf{R})$, $\psi_N(x)=\psi_{\mathbf{R}}(x_{12}+\cdots+x_{n-1,n})$ $(x=(x_{ij})\in N)$
- (π,H_π): irreducible admissible representation of G
- $C^\infty(N\backslash G,\psi_N)=\{f\in C^\infty(G,\mathbf{C})\mid f(xg)=\psi_N(x)f(g),\forall(x,g)\in N\times G\}$
- $\mathcal{I}_{\pi,\psi}:=\{\Phi\mid\Phi\in\mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi,K},C^\infty(N\backslash G,\psi_N))$, $\Phi(v)$ is of moderate growth $(v\in H_{\pi,K})\}$
- $\mathcal{W}(\pi,\psi_{\mathbf{R}})=\{\Phi(v)\mid\Phi\in\mathcal{I}_{\pi,\psi}, v\in H_{\pi,K}\}$: Whittaker model of π

- (π,H_π), $(\pi',H_{\pi'})$: irreducible admissible generic representations of G_{n+1}, G_n
- (τ,V_τ), $(\tau',V_{\tau'})$: irreducible representations of K_n with K_n-embeddings $\varphi:V_\tau\hookrightarrow\mathcal{W}(\pi,\psi_{\mathbf{R}})$, $\varphi':V_{\tau'}\hookrightarrow\mathcal{W}(\pi',\psi_{\mathbf{R}}^{-1})$

Lemma 1. *For $W\in\mathcal{W}(\pi,\psi_{\mathbf{R}})$ and $W'\in\mathcal{W}(\pi',\psi_{\mathbf{R}}^{-1})$ we set*

$$Z(s,W,W'):=\int_{N_n\backslash G_n}W\begin{pmatrix}g&\\&1\end{pmatrix}W'(g)|\det g|^{s-1/2}dg.$$

For $v\in V_\tau$ and $v'\in V_{\tau'}$, we have

$$Z(s,\varphi(v),\varphi'(v'))=\frac{\langle v,v'\rangle}{\dim V_\tau}\sum_{i=1}^{\dim V_\tau}\int_{A_n}\varphi(v_i)\begin{pmatrix}y&\\&1\end{pmatrix}\varphi'(v_i')(y)|\det y|^{s-1/2}\delta(y)^{-1}dy$$

if $\tau'=\widehat{\tau}$ (contragredient of τ), and $Z(s,\varphi(v),\varphi'(v'))=0$ if $\tau'\not\cong\widehat{\tau}$. Here $\{v_i\}$ is a basis of V_τ and $\{v_i'\}$ is its dual basis of $V_{\tau'}=V_{\widehat{\tau}}$ and $\delta(y)=\prod_{i=1}^{n-1}y_i^{i(n-i)/2}$.

Explicit formulas for $GL(2,\mathbf{R})$-Whittaker functions

- $\pi=D_{(\nu,l)}$: discrete series of G_2 $(\nu\in\mathbf{C},l\in\mathbf{Z}_{>0})$
- $\lambda\in\mathbf{Z}_{>0}$
- $(\tau_\lambda^{(2)},V_\lambda^{(2)}=\mathbf{C}v_\lambda\oplus\mathbf{C}v_{-\lambda})\in\widehat{K_2}$ with K_2-action $\tau_\lambda^{(2)}\left(\begin{smallmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{smallmatrix}\right)v_q=e^{\sqrt{-1}q\theta}v_q$, $\tau_\lambda^{(2)}\left(\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right)v_q=v_{-q}$ $(q\in\{\pm\lambda\})$
- $\widehat{K_2}=\{1,\det\}\cup\{\tau_\lambda^{(2)}\mid\lambda\in\mathbf{Z}_{>0}\}$
- $D_{(\nu,l)}|_{K_2}\cong\bigoplus_{i=0}^\infty\tau_{l+1+2i}^{(2)}$

Proposition 2. *There exsits a K_2-embedding $\varphi:V_{l+1}^{(2)}\to\mathcal{W}(D_{(\nu,l)},\psi_{\mathbf{R}}^{\varepsilon})$ $(\varepsilon\in\{\pm1\})$ whose radial part is given by*

$$\varphi(v_{\varepsilon'(l+1)})(\mathrm{diag}(y_1y_2,y_2))=\delta_{\varepsilon,\varepsilon'}\cdot y_1^{1/2}y_2^{2\nu}\cdot\frac{1}{2\pi\sqrt{-1}}\int_s\Gamma_{\mathbf{C}}\Big(s+\nu+\frac{l}{2}\Big)y_1^{-s}\,ds,\qquad(\varepsilon'\in\{\pm1\}),$$

where $\int_s$ is a vertical line from $\mathrm{Re}(s)-\sqrt{-1}\infty$ to $\mathrm{Re}(s)+\sqrt{-1}\infty$ with the sufficiently large real part to keep the poles of the integrand on its left.

Explicit formulas for $GL(3,\mathbf{R})$-Whittaker functions

- $\pi=\mathrm{Ind}_{P_{2,1}}(D_{(\nu_1,l)}\boxtimes\chi_{(\nu_2,\delta)})$: generalized principal series of G_3 $(\nu_1,\nu_2\in\mathbf{C},l\in\mathbf{Z}_{>0},\delta\in\{0,1\})$
- $\lambda=(\lambda_1,\lambda_2)$, $\lambda_1\in\mathbf{Z}_{\geq0}$, $\lambda_2\in\{0,1\}$
- $P_\lambda=\{$degree λ_1 homogeneous polynomial in $z_1,z_2,z_3\}$
- K_3 acts on P_λ by $(T_\lambda(k)p)(z_1,z_2,z_3)=(\det k)^{\lambda_2}p((z_1,z_2,z_3)\cdot k)$, $k\in K_3$, $p\in P_\lambda$
- $(\tau_\lambda^{(3)},V_\lambda^{(3)})\in\widehat{K_3}$: quotient representation of T_λ on $V_\lambda^{(3)}=P_\lambda/(z_1^2+z_2^2+z_3^2)P_{\lambda-(2,0)}$
- $S_\lambda=\{\mathbf{m}=(m_1,m_2,m_3)\in(\mathbf{Z}_{\geq0})^3\mid m_1+m_2+m_3=\lambda_1\}$
- $\{u_{\mathbf{m}}^\lambda=$ image of $z_1^{m_1}z_2^{m_2}z_3^{m_3}$ under $P_\lambda\to V_\lambda^{(3)}\mid\mathbf{m}=(m_1,m_2,m_3)\in S_\lambda\}$: generator of $V_\lambda^{(3)}$
- $\{v_q^\lambda:=$ image of $((\mathrm{sgn}\,q)z_1+\sqrt{-1}z_2)^{|q|}z_3^{\lambda_1-|q|}\mid-\lambda_1\leq q\leq\lambda_1\}$: basis of $V_\lambda^{(3)}$
- $\tau_\lambda^{(3)}\begin{pmatrix}\cos\theta&\sin\theta&\\-\sin\theta&\cos\theta&\\&&1\end{pmatrix}v_q^\lambda=e^{\sqrt{-1}q\theta}v_q^\lambda$, $\tau_\lambda^{(3)}(\mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_3))v_q^\lambda=\varepsilon_1^{\lambda_2}\varepsilon_2^{\lambda_2+q}\varepsilon_3^{\lambda_1+\lambda_2+q}v_{\varepsilon_1\varepsilon_2q}^\lambda$ $(\varepsilon_i\in\{\pm1\})$
- $\tau_\lambda^{(3)}|_{K_2}\cong\det^{\lambda_2}\oplus\bigoplus_{i=1}^{\lambda_1}\tau_i^{(2)}$, $\tau_i^{(2)}\ni v_{\pm i}\to v_{\pm i}^\lambda\in\tau_\lambda^{(3)}$: K_2-injection
- $\pi|_{K_3}\cong\tau_{(l+1,\delta)}^{(3)}\oplus\tau_{(l+2,1-\delta)}^{(3)}\oplus\bigoplus_{i=2}^\infty m(i)\tau_{(l+1+i,\delta(i))}^{(3)}$ with $\delta(i)\equiv\delta+i+1\pmod 2$ and $m(i)\geq2$ (multiplicity)

Theorem 3. [1] *There exists a K_3-embedding $\varphi:V_{(l+1,\delta)}^{(3)}\to\mathcal{W}(\pi,\psi_{\mathbf{R}})$ whose radial part is given by*

$$\varphi(u_{\mathbf{m}}^\lambda)(\mathrm{diag}(y_1y_2y_3,y_2y_3,y_3))=(\sqrt{-1})^{m_1-m_3}y_1y_2(y_2y_3)^{2\nu_1+\nu_2}\cdot\frac{1}{(2\pi\sqrt{-1})^2}\int_{s_2}\int_{s_1}\Gamma_{\mathbf{m}}(s_1,s_2)y_1^{-s_1}y_2^{-s_2}\,ds_1ds_2,$$

where

$$\Gamma_{\mathbf{m}}(s_1,s_2)=\frac{\Gamma_{\mathbf{C}}(s_1+\nu_1+\frac{l}{2})\Gamma_{\mathbf{R}}(s_1+\nu_2+m_1)\Gamma_{\mathbf{C}}(s_2-\nu_1+\frac{l}{2})\Gamma_{\mathbf{R}}(s_2-\nu_2+m_3)}{\Gamma_{\mathbf{R}}(s_1+s_2+m_1+m_3)}.$$

- $\pi = \mathrm{Ind}_{P_{2,1}}(D_{(\nu_1,l)} \boxtimes \chi_{(\nu_2,\delta)})$, $\pi' = D_{(\nu',l')}$ with $l \geq l'$
- $V_\tau = V^{(2)}_{l'+1} = \mathbf{C}v_{l'+1} \oplus \mathbf{C}v_{-(l'+1)}$
- $\varphi : V^{(2)}_{l'+1} \hookrightarrow V^{(3)}_{(l+1,\delta)} \to \mathcal{W}(\pi, \psi_{\mathbf{R}})$
- $\varphi' : V^{(2)}_{l'+1} \to \mathcal{W}(\pi', \psi_{\mathbf{R}}^{-1})$

Theorem 4. [2] *For $v, v' \in V^{(2)}_{l'+1}$, we have*

$$Z(s, \varphi(v), \varphi(v')) = .\langle v, v'\rangle(\sqrt{-1})^{2l'-l+1}\Gamma_{\mathbf{C}}(s+\nu_1+\nu'+\frac{l+l'}{2})\Gamma_{\mathbf{C}}(s+\nu_1+\nu'+\frac{l-l'}{2})\Gamma_{\mathbf{C}}(s+\nu_2+\nu'+\frac{l'}{2}).$$

(outline of proof) We have

$$Z(s) \equiv Z(s,\varphi(v),\varphi'(v')) = \frac{\langle v, v'\rangle}{2}\int_0^\infty\int_0^\infty \varphi(v_{l'+1})(\mathrm{diag}(y_1y_2,y_2,1))\varphi'(v_{-(l'+1)})(\mathrm{diag}(y_1y_2,y_2))y_1^{s-3/2}y_2^{2s-1}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

Since $(z_1+\sqrt{-1}z_2)^{l'+1}z_3^{l-l'} = \sum_{j=0}^{l'+1}\binom{l'+1}{j}\sqrt{-1}^{l'+1-j}z_1^jz_2^{l'+1-j}z_3^{l-l'}$, Mellin-inversion implies that

$$Z(s) = \frac{\langle v, v'\rangle}{2}\sqrt{-1}^{2l'-l+1}\Gamma_{\mathbf{C}}(2s+\nu_1+\nu_2+2\nu'+\frac{l}{2})\Gamma_{\mathbf{R}}(2s+2\nu_1+2\nu'+l-l')\sum_{j=0}^{l'+1}\binom{l'+1}{j}$$
$$\times\frac{1}{2\pi\sqrt{-1}}\int_t \frac{\Gamma_{\mathbf{C}}(s-t+\nu_1+\frac{l}{2})\Gamma_{\mathbf{R}}(s-t+\nu_2+j)}{\Gamma_{\mathbf{R}}(3s-t+2\nu_1+\nu_2+2\nu'+j+l-l')}\cdot\Gamma_{\mathbf{C}}(t+\nu'+\frac{l'}{2})\,dt.$$

Substituting $t \to t+j$ and using the formula

$$\sum_{j=0}^{m}\binom{m}{j}\Gamma_{\mathbf{C}}(a+j)\Gamma_{\mathbf{C}}(b-j) = \frac{\Gamma_{\mathbf{C}}(a)\Gamma_{\mathbf{C}}(b-m)\Gamma_{\mathbf{C}}(a+b)}{\Gamma_{\mathbf{C}}(a+b-m)},$$

we know that

$$Z(s) = \frac{\langle v, v'\rangle}{2}\sqrt{-1}^{2l'-l+1}\Gamma_{\mathbf{C}}(2s+\nu_1+\nu_2+2\nu'+\frac{l}{2})\Gamma_{\mathbf{R}}(2s+2\nu_1+2\nu'+l-l')\cdot\frac{\Gamma_{\mathbf{C}}(s+\nu_1+\nu'+\frac{l+l'}{2})}{\Gamma_{\mathbf{C}}(s+\nu_1+\nu'+\frac{l-l'}{2}-1)}$$
$$\times\frac{1}{2\pi\sqrt{-1}}\int_t\frac{\Gamma_{\mathbf{R}}(s-t+\nu_2)\Gamma_{\mathbf{C}}(t+\nu'+\frac{l'}{2})\Gamma_{\mathbf{C}}(s-t+\nu_1+\frac{l}{2}-l'-1)}{\Gamma_{\mathbf{R}}(3s-t+2\nu_1+\nu_2+2\nu'+l-l')}\,dt.$$

From the formulas $\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$ (duplation formula) and

$$\frac{1}{4\pi\sqrt{-1}}\int_t\frac{\Gamma_{\mathbf{R}}(t+a)\Gamma_{\mathbf{R}}(t+b)\Gamma_{\mathbf{R}}(t+c)\Gamma_{\mathbf{R}}(-t+d)\Gamma_{\mathbf{R}}(-t+e)}{\Gamma_{\mathbf{R}}(t+a+b+c+d+e)}\,dt$$
$$= \frac{\Gamma_{\mathbf{R}}(a+d)\Gamma_{\mathbf{R}}(b+d)\Gamma_{\mathbf{R}}(c+d)\Gamma_{\mathbf{R}}(a+e)\Gamma_{\mathbf{R}}(b+e)\Gamma_{\mathbf{R}}(c+e)}{\Gamma_{\mathbf{R}}(a+b+d+e)\Gamma_{\mathbf{R}}(a+c+d+e)\Gamma_{\mathbf{R}}(b+c+d+e)}$$

(Barnes' second lemma), we can get the assertion. □

- Barnes' frist lemma:

$$\frac{1}{4\pi\sqrt{-1}}\int_t\Gamma_{\mathbf{R}}(t+a)\Gamma_{\mathbf{R}}(t+b)\Gamma_{\mathbf{R}}(-t+c)\Gamma_{\mathbf{R}}(-t+d)\,dt = \frac{\Gamma_{\mathbf{R}}(a+c)\Gamma_{\mathbf{R}}(a+d)\Gamma_{\mathbf{R}}(b+c)\Gamma_{\mathbf{R}}(b+d)}{\Gamma_{\mathbf{R}}(a+b+c+d)}$$

Remark 1. The case $l' > l$:

- $E_{ij} \in \mathfrak{gl}(3,\mathbf{R})$: matrix unit
- $\lambda \in \mathbf{Z}_{>0}$, $\mathcal{D}_\lambda := \mathbf{C}\text{-span}\{E_\lambda = (E_{23}-\sqrt{-1}E_{13})^\lambda, E_{-\lambda} = (E_{23}+\sqrt{-1}E_{13})^\lambda\} \subset U(\mathfrak{gl}(3,\mathbf{C}))$
- $\mathcal{D}_\lambda \cong V^{(2)}_\lambda$ as K_2-module (adjoint action on $\mathcal{D}_\lambda$) via $E_{\pm\lambda} \leftrightarrow v_{\pm\lambda}$
- $\varphi : V^{(2)}_{l'+1} \hookrightarrow V^{(2)}_{l'-l} \otimes V^{(2)}_{l+1} \to \mathcal{W}(\pi, \psi_{\mathbf{R}})$, $\varphi' : V^{(2)}_{l'+1} \to \mathcal{W}(\pi', \psi_{\mathbf{R}}^{-1})$

References

[1] Tadashi Miyazaki, Whittaker functions for generalized principal series representations of SL$(3,\mathbf{R})$. Manuscripta Math., **128** (2009), 107–135.

[2] Miki Hirano, Taku Ishii and Tadashi Miyazaki, The archimedean zeta integrals for $GL(3) \times GL(2)$. Proc. Japan Acad., **92**. Ser. A (2016), 27–32.

Faculty of Science and Technology, Seikei University, 3-3-1 Kichijoji-Kitamachi, Musashino, Tokyo, 180-8633, Japan
E-mail address: ishii@st.seikei.ac.jp