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First talk

I. A construction of the Néron-Tate height pairing on abelian varieties by S. Bloch [4]

II. Several constructions of the p-adic heights

III. Comparison of p-adic heights

p-adic height ... $\mathbb{Q}_p$-valued height related to p-adic BSD.

1. The "usual" construction of Néron-Tate canonical height on abelian var $A/$number field

The ht on $\mathbb{P}^n$ $\rightsquigarrow$ A naive ht on A by $A \hookrightarrow \mathbb{P}^n$ e.g.

naive Néron local ht. $\rightsquigarrow$ (canonical) $\sum \langle\ ,\ \rangle_v$ Néron local ht pairing

product formula

Tate's 2-power technique

Néron-Tate can. ht pairing

local ht pairing has "characterization".

2. Construction by Bloch [4]

<u>Outline</u> $A/\mathbb{Q}$: abel. var.

$|\ |_v : v = p, \infty$ normalize $\prod_v |x|_v = 1$

so that $x \in \mathbb{Q}^\times$

Want to define

$$A^\vee(\mathbb{Q}) \times A(\mathbb{Q}) \longrightarrow \mathbb{R}$$

dual abelia var. $(x, y) \longmapsto \langle x, y \rangle_{\mathrm{Bloch}}$

Regard

<u>Step 1</u> $x \in A^\vee(\mathbb{Q}) = \mathrm{Ext}_{\mathbb{Q}, gp}(A, \mathbb{G}_m)$

i.e.

$$\exists\ 0 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}_x \longrightarrow A \longrightarrow 0$$

<u>Step 2</u> $v = p, \infty$

construct $l_{x,v} : \mathcal{G}_x(\mathbb{Q}_v) \longrightarrow \mathbb{R}$

hom. cont. extending $\log |\ |_v : \mathbb{Q}_v^\times \longrightarrow \mathbb{R}$

$(\mathbb{Q}_\infty = \mathbb{R})$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Q}_v^\times \arrow[r] \arrow[d, "\log|\ |_v"'] & \mathcal{G}_x(\mathbb{Q}_v) \arrow[r] \arrow[dl, "\exists!\, l_{x,v}"] & A(\mathbb{Q}_v) \arrow[r] & 0 \\ & \mathbb{R} & & & \end{tikzcd}$$

uniqueness of $l_{x,v}$ follows from

$A(\mathbb{Q}_v)$ compact. $\mathbb{R}$ has no non-trivial

compact subgp

By Bourbaki (Integration), it always exists

(Oesterlé [25])

For $a \in \mathcal{G}_x(\mathbb{Q})$, $l_{x,v}(a) = 0$ for almost all v

$$l_x = \sum_v l_{x,v} : \mathcal{G}_x(\mathbb{Q}) \longrightarrow \mathbb{R}$$

Step 3 $\langle x, y \rangle_{\mathrm{Bloch}} = l_x(\hat{y})$

where $\hat{y}$ is any lift of y by $\mathcal{G}_x(\mathbb{Q}) \to A(\mathbb{Q}) \to 0$

$l_x(\mathbb{Q}^\times) = 0$ welldefined.

(bilinear not clear...)

<u>local height pairing</u>

$$\langle\ ,\ \rangle_{\mathrm{Bloch}, v} : \big(\mathrm{Div}^0(A)(\mathbb{Q}_v) \times \underbrace{Z_0(A)^0}_{\text{zero cycle, degree } 0}(\mathbb{Q}_v)\big) \longrightarrow \mathbb{R}$$

disjoint support

For $x \in \mathrm{Div}^0(A)(\mathbb{Q}_v)$,

$$\exists\ 1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}_x \longrightarrow A \longrightarrow 0$$

$\exists$ tautological section

$$\underbrace{A \setminus |x|}_{|x|:\ \text{support}} \xrightarrow{s_x} \mathcal{G}_x$$

up to constant multiplication

$$\langle x, y \rangle_{\mathrm{Bloch}, v} := l_{x,v}(s_x(y))$$

$y = \sum n_i P_i$ $\qquad A(\overline{\mathbb{Q}}_v) \ni P_i$

$\sum n_i P_i$: $\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$-fixed.

$$s_x(y) := \prod_i s_x(P_i)^{n_i}$$

well-defined by $\sum n_i = 0$

<u>Thm (Bloch)</u> $\langle\ ,\ \rangle_{\mathrm{Bloch},v} = \langle\ ,\ \rangle_{NT,v}$

Néron-Tate

$$\langle\ ,\ \rangle_{\mathrm{Bloch}} = \sum_v \langle\ ,\ \rangle_{\mathrm{Bloch},v}$$

$$\langle\ ,\ \rangle_{\mathrm{Bloch}} = \langle\ ,\ \rangle_{NT} \quad \text{global}$$

(Proof) Check the characterization of local Néron-Tate ht.

3. Bloch's motivation

Give an interpretation of full BSD formula as Tamagawa number formula.

Suppose $r = \operatorname{rank} \check{A}(\mathbb{Q})$

Fix $A^\vee(\mathbb{Q}) = \underset{\text{free}}{N} \oplus \check{A}(\mathbb{Q})_{\text{tor}}$

$$\exists\ 1 \longrightarrow N_m \longrightarrow \mathcal{G}_N \longrightarrow A \longrightarrow 0$$

$$\begin{tikzcd} N_m \arrow[d, "\log"'] & \mathcal{G}_N \arrow[dl, "\exists!\, l_{N,v}"] \\ \mathbb{R}^r & \end{tikzcd}$$

$\exists N \cong \mathbb{G}_m^r$ non-canonically

Then, $L(\mathcal{G}_N, s) = L(A,s)L(\mathbb{G}_m^r, s)$

$= L(A,s)\,\zeta(s)^r \leftarrow$ Riemann zeta

(weak) BSD for $A \Longleftrightarrow L(\mathcal{G}_N, 1) \neq 0, \infty$

BSD $\Rightarrow L(\mathcal{G}_N, 1) \neq 0, \infty$, $\mathcal{G}_N(\mathbb{Q}) \subset \mathcal{G}(\mathbb{A}_{\mathbb{Q}})$ discrete cocompact.

$\Rightarrow$ Tamagawa number $\tau(\mathcal{G}_N)$ is defined.

<u>Tamagawa number conj</u>

$$\tau(\mathcal{G}_N) = \frac{\# \mathrm{Pic}(\mathcal{G}_N)_{tor}}{\# Ш(\mathcal{G}_N)}$$

$$\frac{\prod_{v \leq \infty} C_{v,A} \; \# A(\mathbb{Q})_{tor}^{-1} \cdot \mathrm{Reg}(\mathrm{Bloch})}{L(\mathcal{G}_N, 1)}$$

$\leftarrow \det \langle P_i, P_j \rangle_{\mathrm{Bloch}}$, $1 \leq i, j \leq r$; (P_i): generator of $A(\mathbb{Q})/A(\mathbb{Q})_{tor}$

full BSD $\Leftrightarrow$ Tam conj for $\mathcal{G}_N$

☆ "Bloch's construction" can be generalized to Galois rep?

$H^1_f(V^*(1)) \times H^1_f(V) \longrightarrow \mathbb{R}$?

$\rightsquigarrow$ Bloch-Kato Tamagawa number conj

can be also p-adified

4. $A^\vee = \mathrm{Ext}_{gp}(A, \mathbb{G}_m)$

$\mathrm{Div}^0(A) =$ "$\mathrm{Ext}\ \mathrm{rig}$" $(A, \mathbb{G}_m)$

section + ext.

K : field (e.g. $K = \mathbb{Q}, \mathbb{Q}_v$)

A/K : abel. var.

$D \in \mathrm{Div}^0(A)(K)$

$D \in \mathrm{Div}^0(A) \iff m^*D - p_1^*D - p_2^*D$ is principal on $A \times A$.

alg. eq. to 0

(where $m : A \times A \longrightarrow A$ summation

$p_i : A \times A \longrightarrow A \quad (i=1,2)$ i-th projection)

$\iff T_x^*D - D$ is principal

for $\forall x \in A(\bar{K})$, T_x : translation by x

Construction of

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathcal{G}_D \arrow[r] & A \arrow[r] & 0 \\ & & & A \setminus D \arrow[u, phantom, "\cup" rotate=90] \arrow[ul, bend left, "s_D"] & \end{tikzcd}$$

$\exists$ two methods

- theory of birational gp [5] + [30]
- theta gp