Kobayashi 2012年4月5日

K : field $= (\mathbb{Q}, \mathbb{Q}_p, \mathbb{R}, \mathbb{C})$

A/K : abel. var $\quad \underset{\text{origin.}}{e_A} \in A(K) \quad \left(\begin{array}{l} A/S \text{ : Abel. scheme} \\ e_A^* \mathcal{O}_A(D) \simeq \mathcal{O}_S \end{array} \right)$

$D \in \mathrm{Div}^0(A)(K)$

Aim

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathcal{G}_D \arrow[r] & A \arrow[r] & 0 \\ & & & A \setminus |D| \arrow[u, phantom, "\cup"] \arrow[ul, bend left, "s_D"] & \end{tikzcd}$$

invertible $\mathcal{L} = \mathcal{O}_A(D) \xleftrightarrow[\text{identify}]{} L$: line bundle on A

$$\begin{tikzcd} L \arrow[d] \\ A \arrow[u, bend right, "\text{0-section}"'] \end{tikzcd}$$

$\mathcal{G}_D = L \setminus$ 0-section

Fix trivialization, $e_A{}^* \mathcal{L} \cong \mathcal{O}_{\mathrm{Spec}\,K} = K$

$$\begin{array}{ccc} e_A^* L & \cong & \mathbb{A}^1_K \\ \Psi & & \Psi \\ e_D & \leftarrow\!\shortmid & 1 \end{array}$$

$$\begin{array}{ccc} \mathcal{G}_D(K) & \longrightarrow & A(K) \\ \Psi & & \Psi \\ e_D & \longmapsto & e_A \end{array}$$

We define a group law on $\mathcal{G}_D$ so that e_D becomes the unit element.

$S :=$ scheme$/K$

X/K : scheme$/K \qquad X_S = X \times_K S$

$$\underline{\mathrm{Aut}}(L/A)(S) := \left\{ (p, \tau_p) \;\middle|\; \begin{array}{c} p \in A(S) \\ \begin{tikzcd} L_S \arrow[r, "\tau_p"] \arrow[d] & L_S \arrow[d] \\ A_S \arrow[r, "T_p"'] & A_S \end{tikzcd} \text{ auto} \end{array} \right\}$$

T_P : translation by P

$$= \left\{ (P, \tau_P) \;\middle|\; \begin{array}{l} P \in A(S) \\ T_P^* \mathcal{L}_S \underset{\tau_P}{\cong} \mathcal{L}_S \end{array} \right\}$$

<u>Prop (Mumford, Abelian varieties)</u>

Functor $(Sch/_K) \longrightarrow (group)$

$$S \longrightarrow \underline{Aut}(L/A)(S)$$

is representable by $\mathcal{G}_D$. ⌟

We have an exact seq. of Zariski sheaves.

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}_D \longrightarrow A \longrightarrow 0$$

On S-valued pts, this gives

$$0 \longrightarrow \Gamma(S, \mathcal{O}_S^\times) \longrightarrow \underline{Aut}(L/A)(S) \longrightarrow A(S)$$

$$A(S) = \left\{ P \in A(S) \;\middle|\; \begin{array}{l} \overset{\text{always hold}}{\downarrow} \\ T_P^* \mathcal{L} \otimes \mathcal{L}^{-1} \text{ is pull-back of an element of } Pic(S) \end{array} \right\}$$

$$A(S) \supset \{P \in A(S) \mid T_P^* \mathcal{L}_S \cong \mathcal{L}_S\} \longrightarrow 0$$

$$\underline{Aut}(L/A)(S) \longrightarrow \{P \in A(S) \mid T_P^* \mathcal{L}_S \cong \mathcal{L}_S\} \longrightarrow 0$$

In particular, if $Pic(S) = \{0\}$

$$0 \longrightarrow \mathbb{G}_m(S) \longrightarrow \mathcal{G}_D(S) \longrightarrow A(S) \longrightarrow 0$$

<u>Rem</u> $\Gamma(A_S, \mathcal{O}_{A_S}^\times) \underset{\underset{\text{proper}}{\uparrow}}{=} \Gamma(S, \mathcal{O}_S^\times) \Rightarrow$ exact.

- $\mathcal{G}_D$ is comm. since $Map(A, \mathbb{G}_m) = \{\text{constant}\}$

  $\mathbb{G}_m$ is in the center ~~of~~ in $\mathcal{G}_D$

$$A \times A \longrightarrow \mathbb{G}_m$$

$$(x, y) \underset{\text{trivial}}{\longmapsto} \hat{x}\hat{y}\hat{x}^{-1}\hat{y}^{-1} \qquad \hat{x}, \hat{y} \text{ lift of } x, y$$

Proof We have the following fundamental isom.

$$\underline{\mathrm{Aut}}(L/A)(S) \xrightarrow[\iota(S)]{\sim} \mathcal{G}_D(S)$$

$$\alpha \longmapsto \alpha(e_{D,S})$$

$$e_{D,S} : S \longrightarrow S \times K \xrightarrow{\mathrm{id} \times e_D} S \times \mathcal{G}_D$$

inj : follows from $H^0(A_S, \mathcal{O}_{A_S}^{*}) = H^0(S, \mathcal{O}_S^{*})$

surj : take $\phi \in \mathcal{G}_D(S)$, $\phi \mapsto p \in A(S)$ $\qquad S = {}^{\exists}\bigcup S_i$

$$T_p^{*}\mathcal{L}_{S_i} \overset{f_i}{\cong} \mathcal{L}_{S_i}$$

modify f_i so that $\iota(S_i)((p, f_i)) = \phi|_{S_i}$

$$\xrightarrow{\text{patch}} T_p^{*}\mathcal{L}_S \overset{f}{\cong} \mathcal{L}_S$$

$$\iota(S)((p, f)) = \phi$$

Cor $\mathcal{G}_D(\bar{K}) = \{(a, f(x)) \in A(\bar{K}) \times K_{A_{\bar{K}}} \mid D - T_a^{*}D = (f(x))\}$ (funct. field of $A_{\bar{K}}$)

group law $(a, f)\cdot(b, g) = (a+b, T_a^{*}g \cdot f)$

Local description of $\mathcal{G}_D$

Fix a rational funct. f_D on $A \times A$

s.t. $(f_D) = p_1^{*}D + p_2^{*}D - m^{*}D$

$U = A \setminus |D|$

$$\mathcal{G}_{D|U}(\bar{K}) \cong U(\bar{K}) \times \bar{K}^{\times}$$

$$(a, f(x)) \mapsto (a, \underbrace{f(x)\cdot f_D(x, a)^{-1}}_{\text{const.}})$$

(birational) gp law on $U(\bar{K}) \times \bar{K}^\times$

$$(a,k)\cdot(b,l) = (a+b,\ f_D(a,b)\,kl)$$
$$(\text{if } a, b, a+b \notin D)$$

$$S_D : U(\bar{K}) \longrightarrow U(\bar{K}) \times \bar{K}^\times \subset \mathcal{G}_D(\bar{K})$$
$$a \longmapsto (a, 1)$$

In general, $D = \{(U_i, f_i)\}_i$ Cartier div.

$$\mathcal{G}_{D|U_i}(\bar{K}) \cong U_i(\bar{K}) \times \bar{K}^\times$$
$$(a, f(x)) \longmapsto (a,\ f(x)\, g_{D,i}^{-1}\overbrace{(x,a)\, T_a^* f_i \cdot f_i^{-1}}^{\text{const}})$$

$$g_{D,i} = f_D \cdot \underset{\substack{\Vert\\ \text{no zero \& pole on } U_i}}{m^* f_i} \cdot p_1^* f_i^{-1} \cdot p_2^* f_i^{-1}$$

gp law on $U_i(\bar{K}) \times \bar{K}^\times$

$$(a,k)(b,l) = (a+b,\ g_{D,i}(a,b)\,kl)$$
$$(a, b, a+b \notin D)$$

$$S_D : U_i(\bar{K}) \setminus |D| \longrightarrow U_i(\bar{K}) \times \bar{K}^\times \subset \mathcal{G}_D(\bar{K})$$
$$a \longmapsto (a, f_i(a))$$

<u>Example</u> $A = E/\mathbb{Q}_p$ ellip. curve good.
$D = (P) - (Q)$ $\quad P, Q \in E_1(\mathbb{Q}_p) = \hat{E}(p\mathbb{Z}_p)$

$\hat{\mathcal{G}}_D = \hat{E} \times \hat{\mathbb{G}}_m$ gp law.

$$(a,k)\cdot(b,l) := \left(a \underset{\hat{E}}{\oplus} b,\ (\hat{g}_D(a,b) - 1) \underset{\hat{\mathbb{G}}_m}{\oplus} k \underset{\hat{\mathbb{G}}_m}{\oplus} l\right)$$

$$\hat{g}_D(s,t) = \frac{\hat{\sigma}_0(s\ominus P)\,\hat{\sigma}_0(t\ominus P)\,\hat{\sigma}_0(s\oplus t\ominus Q)\,\hat{\sigma}_0(-Q)}{\hat{\sigma}(s\oplus t\ominus P)\,\hat{\sigma}_0(s\ominus Q)\,\hat{\sigma}_0(t\ominus Q)\,\hat{\sigma}_0(-P)} \quad ($$

$\in \mathbb{Z}_p[[S,T]]^\times \qquad \hat{g}_D(0,0)=1$

$\hat{\sigma}_0(t) = \sigma(\lambda(t))/t \qquad t = -\frac{x}{y}$ $\hat{E}$ parameter

σ : (formal) Weierstraß

$\lambda(t)$: log of $\hat{E}$

5. The ext. $l_{D,v} : \mathcal{G}_D(\mathbb{Q}_v) \longrightarrow \mathbb{R}$

extending $\log|\ |_v : \mathbb{Q}_v^\times \to \mathbb{R}$

$\exists$ integral model of $\mathcal{G}_D/\mathbb{Q}_v$. $\mathcal{O}_v = \begin{cases} \mathbb{Z}_p \\ \mathbb{R} \text{ or } \mathbb{C} \end{cases}$

as before

$v \neq \infty$,

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{G}_m(\mathcal{O}_v) & \to & \mathcal{G}_D(\mathcal{O}_v) & \to & A^\circ(\mathcal{O}_v) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{G}_m(\mathbb{Q}_v) & \to & \mathcal{G}_D(\mathbb{Q}_v) & \to & A(\mathbb{Q}_v) & \to & 0 \\ & & \downarrow{\scriptstyle \log|\ |_v} & \swarrow{\scriptstyle l_{D,v}} & & & \downarrow & & \\ & & \mathbb{R} & & & & \text{finite} & & \end{array}$$

$\log|\mathcal{O}_v^\times|_v = 0$

<u>$v = \infty$</u> $\quad \mathbb{G}_m(\mathcal{O}_v),\ \mathcal{G}_D(\mathcal{O}_v),\ A^\circ(\mathcal{O}_v)$

$\quad$ '' $X(\mathcal{O}_v)$

$\longrightarrow X(\mathbb{R})$ max. compact gp

$X(\mathbb{C})$ max compact gp

similarly

Other methods

Bourbaki (Integral) remarked by Oesterlé

· Tate's method 2-power method.
(Lang's book) [15]

6. Néron local height pairing

$\exists ! \quad \langle \, , \, \rangle_v : (\mathrm{Div}^0 A(\mathbb{Q}_v)) \times \underset{\substack{\uparrow \\ \text{0-cycle, deg}=0}}{Z_0(A)^0(\mathbb{Q}_v)} \xrightarrow[\text{dis supp.}]{} \mathbb{R}$

s.t i) bilinear (if it makes sense)

ii) If $D = (f)$ principal

$\langle (f), \alpha \rangle_v = -\log |f(\alpha)|_v$

$\alpha = \sum n_i P_i \in \mathrm{Div}^0 A(\overline{\mathbb{Q}_v})^{\mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v)}$

$f(\alpha) := \prod_i f(P_i)^{n_i}$

iii) Any finite $A \longrightarrow A$

$\langle \phi^* D, \alpha \rangle_v = \langle D, \phi_* \alpha \rangle_v$

iv) $\forall D, x_0 \in A(\mathbb{Q}_v) \setminus |D|$

$A(\mathbb{Q}_v) \setminus |D| \longrightarrow \mathbb{R}$

$x \longmapsto \langle D, (x) - (x_0) \rangle$ is continuous

call $\underbrace{\text{Néron axiom}}_{\text{(i)}\sim\text{(iv)}}$

$\langle D, \alpha \rangle_{\mathrm{Bloch}, v} = l_{D,v}(S_D(\alpha))$
satisfies Néron axiom.

ii) $\sim \sum_v \langle \, , \, \rangle_v$ is defined on $\check{A}(\mathbb{Q}) \times A(\mathbb{Q}) \to \mathbb{R}$