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補足 Reference

Mazur-Tate, canonical height pairings via biextension[s]

in Arith. Geo. Prog. Math. 1983.

choice of log

group law on $\mathcal{G}_D(\bar{K})$ $(a, f(x))(b, g(x)) = (a+b, T_a^* g \cdot f(x))$

II. Several constructions of p-adic height pairings

$A/\mathbb{Q}$ : Abel. var. $p$ : good prime

For a finite place $v$ of $\mathbb{Q}$, define $\ell_v : \mathbb{Q}_v^\times \longrightarrow \mathbb{Q}_p$

$$\ell_v(x) := \begin{cases} -\log_p |x|_v & (v \neq p) \\ -\log_p x & (v = p) \end{cases}$$

$\log_p p = 0$, $|\ell|_\ell = \frac{1}{\ell}$ · choice of log · choice of Hodge f.l.

Note $\sum_{\text{finite places}} \ell_v(x) = 0$ if $x \in \mathbb{Q}^\times$

Rem Choice of $\ell_v$ corresponds to the choice of $\mathbb{Z}_p$-ext.

(c.f remarked by [MT]) Our choice here is related to the cyclotomic $\mathbb{Z}_p$-ext.

We want to define

$\langle\ ,\ \rangle_p : A^\vee(\mathbb{Q}) \times A(\mathbb{Q}) \longrightarrow \mathbb{Q}_p$ p-adic ht pairing

+ p-adic local ht pairing

$D \in \mathrm{Div}^0 A(\mathbb{Q})$ $v$ : finite place

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q}_v^\times & \longrightarrow & \mathcal{G}_D(\mathbb{Q}_v) & \longrightarrow & A(\mathbb{Q}_v) & \longrightarrow & 0 \\ & & \downarrow \ell_v & \swarrow \ell_{D,v} & & & & & \\ & & \mathbb{Q}_p & & & & & & \end{array}$$

$\ell_{D,v}$ — want to construct this

Then p-adic ht is constructed as in Bloch's method.

$\underline{v \neq p}$ $\longrightarrow$ $\exists!\, l_{D,v}$ constructed as before

$\boxed{v = p}$ uniqueness $\mathrm{Hom}_{\mathrm{cont}}(A(\mathbb{Q}_v), \mathbb{Q}_p) = 0$

$\searrow$ $\exists$ several constructions of $l_{D,v}$ and several extensions $\longrightarrow$ $\mathrm{Hom}_{\mathrm{cont}}(A(\mathbb{Q}_p), \mathbb{Q}_p) \neq 0$

(in ordinary case, Néron-axiom + extra boundedness cond. for the cyclo. $\mathbb{Z}_p$-ext. characterize a unique "$l_{D,v}$")

## Construction ①

For a splitting $N$ of the Hodge filtration

(i.e. a complementary $\mathbb{Q}_p$-vect. sp $N$ s.t.

$$H^1_{dR}(A/\mathbb{Q}_p) = N \oplus \mathrm{Fil}^0 \text{)}$$

we can construct $l_{D,v} = l_{D,v,N}$ as follows

It suffices to construct

$$S_{D,p} : A(\mathbb{Q}_p) \otimes \mathbb{Q}_p \longrightarrow \mathcal{G}_D(\mathbb{Q}_p) \otimes \mathbb{Q}_p$$

$$0 \longrightarrow H^1_{dR}(A/\mathbb{Q}_p) \longrightarrow H^1_{dR}(\mathcal{G}_D/\mathbb{Q}_p) \longrightarrow H^1_{dR}(\mathbb{G}_m/\mathbb{Q}_p) \longrightarrow 0$$

$\circlearrowleft \varphi$: Frobenius

Eigen polynomials $\varphi$ for $H^1_{dR}(A/\mathbb{Q}_p)$, $H^1_{dR}(\mathbb{G}_m/\mathbb{Q}_p)$ are coprime

$\Rightarrow$ split as $\varphi$-module (Filtration is not preserved)

$$H^1_{dR}(\mathcal{G}_D/\mathbb{Q}_p) = H^1_{dR}(A/\mathbb{Q}_p) \oplus H^1_{dR}(\mathcal{G}/\mathbb{Q}_p)^{\varphi=1}$$

$$H^1_{dR}(\mathcal{G}_D/\mathbb{Q}_p)^{\varphi=1} \xrightarrow{\sim} H^1_{dR}(\mathbb{G}_m/\mathbb{Q}_p)$$

Fix $N$ s.t $H^1_{dR}(A/\mathbb{Q}_p) = N \oplus \mathrm{Fil}$

$$N_D := N \oplus H^1_{dR}(\mathcal{G}_D/\mathbb{Q}_p)^{\varphi=1}$$

$$\mathrm{Fil}^1 H^1_{dR}(\mathcal{G}_D/\mathcal{O}_p) \longrightarrow H^1_{dR}(\mathcal{G}_D/\mathcal{O}_p) \longrightarrow H^1_{dR}(\mathcal{G}_D/\mathcal{O}_p)/N_D$$

$$\cong H^1_{dR}(A/\mathcal{O}_p)/N$$

$$\cong \mathrm{Fil}^1 H^1_{dR}(A/\mathcal{O}_p) = L_A \quad \text{inv. diff.}$$

Then $S_{D,p} : A(\mathcal{O}_p) \otimes \mathcal{O}_p = \mathrm{Hom}_{\mathcal{O}_p}(L_A, \mathcal{O}_p)$

$$\xrightarrow[\text{by the above}]{} \mathrm{Hom}_{\mathcal{O}_p}(L_D, \mathcal{O}_p) = \mathcal{G}_D(\mathcal{O}_p) \otimes \mathcal{O}_p$$

<u>Rem</u> 1) If $p$ is ordinary, a can. choice of $N$

"unit root space"

$\exists$ "canonical" $p$-adic height.

2) $p$-adic $L$-function also depends on the choice "$N$"

ellip. curve case

ordinary $\exists!$ "bounded" $p$-adic $L$ $\longleftrightarrow$ unit root space

super singular $\alpha$ : a root of $X^2 - a_p X + p = 0$

$\mathcal{L}_p(E, \alpha, s) \longleftrightarrow \varphi$-eigen space with eigen value $\alpha$.

## <u>Construction ②</u>

For a normalization $N$ of the $p$-adic theta ④

$\rightsquigarrow \exists\, l_{D,p} = l_{D,p,N}$

(Zarhin [35], $\xrightarrow[\text{Nékovar [22]}]{}$ Galois rep.)

(Néron, $\underset{\text{Mazur-Tate's sigma.}}{\text{MTT}}$, BP[1])

$$\vartheta(z) = \sigma(z) \exp\left(\tfrac{a}{2} z^2\right) \quad \text{normalization} \quad a \in \mathcal{O}_p$$

$P, Q \in E_1(\mathcal{O}_p) = \hat{E}(p\mathbb{Z}_p)$, $D = (P) - (Q)$

$\log_{D,p,N} = \log_p \hat{\vartheta}_D^0 + \log_{\hat{\mathbb{G}}_m}$, $\vartheta_D = \vartheta(z-P) / \vartheta(z-Q)$

$\vartheta^0 = \theta(\lambda(t))/t$

$\left(\frac{d}{dz}\right)^2 \log \vartheta = \eta + a\omega$

$N = \mathcal{O}_p(\eta + a\omega)$ 1

$\eta = \wp(z)dz = x\frac{dx}{y}$ $\omega = dz = \frac{dx}{y}$

$$\langle D, \mathfrak{a} \rangle_{P,P,N} = \log_p \frac{\vartheta(R-P)\vartheta(S-Q)}{\vartheta(R-Q)\vartheta(S-P)} \left( =: \ell_{D,p,N}(S_D(\mathfrak{a})) \right)$$

p-adic local ht at p

Explicit formula (global p-adic ht) $P \in E(\mathbb{Q})$ s.t. $P \in E_1(\mathcal{O}_p)$, $P \in E^{ns}(\mathcal{O}_\ell)$

correct

$$h_{p,N}(P) = \langle P, P \rangle_{p,N} = \underbrace{-}_{\text{MTT is false}} \log_p \left( \vartheta(P)^2 / \mathrm{den}\, x(P) \right)$$

denominator

$\vartheta$ normalization $\hookrightarrow N$

$\operatorname{div} \vartheta = [0]$

+1

<u>Rem 1</u> $E \longrightarrow E^\vee$ $P \mapsto (-P) - (0)$

principal polarization

$\left( E \longrightarrow E^\vee,\ P \mapsto (P) - (0) \text{ in Silverman } \cancel{\text{false}} \right)$

The sign will change

2 the sign of MTT seems wrong

$\updownarrow$ opposite

Mazur-Tate-Stein

Bernardi-Perrin Riou is also false.

<u>ordinary</u>

unit root space $\longrightarrow$ Mazur-Tate $\sigma$-function

characterize $e_2^*$

|

$\sigma_{MT} = \sigma(z)\exp\left(-\frac{e_2^*}{2}z^2\right) \in \mathbb{Z}_p[[t]]$ — integral

$e_2^*$ : Katz's p-adic Eis. $e_2^* \in \mathbb{Z}_p$ $t = -\frac{x}{z}$

## Construction 3 

Universal norm construction (Schneider)

**ordinary** $X/\mathbb{Q}_p$ comm. gp scheme

$\mathbb{Q}_\infty/\mathbb{Q}$ : cyc $\mathbb{Z}_p$-ext. $\mathbb{Q}_{p,n}$ $n$-th layer

$$NX(\mathbb{Q}_p) = \bigcap_n \mathrm{Norm}_{\mathbb{Q}_{p,n}/\mathbb{Q}_p} X(\mathbb{Q}_{p,n})$$ ← universal norm

$$\begin{tikzcd}
0 \arrow[r] & N\mathbb{G}_m(\mathbb{Q}_p) \arrow[r] \arrow[d] & N\mathcal{G}_D(\mathbb{Q}_p) \arrow[r] \arrow[d] & NA(\mathbb{Q}_p) \arrow[r] \arrow[d] & 0 \\
 & \mathbb{G}_m(\mathbb{Q}_p) \arrow[r] \arrow[d, "l_p"'] & \mathcal{G}_D(\mathbb{Q}_p) \arrow[r] \arrow[dl, "\exists l_{D,p}"] & A(\mathbb{Q}_p) \arrow[r] \arrow[d] & 0 \\
 & \mathbb{Q}_p & & \text{finite} & \text{ordinary} \arrow[l]
\end{tikzcd}$$

$l_p | N\mathbb{G}_m(\mathbb{Q}_p) = 0$

LCF, $\log_p P = 0$

( ← Nekovar Heegner cycle mistakes
II Prop. 5.6 X
I 6.6 Corollary
"the sign" )

**Comparison**

1 ⟺ 3 Coleman power series

$$NX(\mathbb{Q}_p) = \hat{X}(\mathrm{Spf}\, \mathbb{Z}_p[[T]])^{P(\varphi)=0} / (\gamma - 1)$$ $\gamma \in \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ generator

$P$: Euler polynomial at "$\varphi$"

$X : \hat{\mathbb{G}}_m$ $P(\varphi) = 0 \Rightarrow \mathrm{Norm} = 1$

$$NX(\mathbb{Q}_p) \underset{\text{P-R}}{\leftrightarrows} \hat{X}(\mathrm{Spf}\, \mathbb{Z}_p[[T]]) / (\gamma - 1)$$

Coleman, P-R, Kobayashi Ohta
(ordinary) (non-ordinary)

$$= \hat{X}(\mathrm{Spf}\ \mathbb{F}_p[[T]])$$

$$= \mathrm{Hom}_{\varphi}(M_{\hat{X}}, CW(\mathbb{F}_p[[T]]))$$

Dieudonné for $\hat{X}$

$$0 \longrightarrow \left(\quad\right)_{p}^{\text{slope}=0} \longrightarrow H^1_{dR}(X/\mathcal{O}_p) \longrightarrow H^1(\hat{X}/\mathcal{O}_p) \longrightarrow 0$$

unit root space $\qquad\qquad$ slope $>0$

① $\Leftrightarrow$ ② explicit calculation
using Dieudonné theory of formal group.