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On the Stickelberger elements and annihilation results for class groups

k: totally real field

K: CM-field s.t K/k finite abelian.

$$\vartheta_{K/k} = \sum_{\sigma \in \mathrm{Gal}(K/k)} \zeta(0,\sigma)\sigma^{-1} \in \mathbb{Q}[\mathrm{Gal}(K/k)]$$

Thm (Deligne-Ribet, Cassou-Noguès, Barsky)

$$\mathrm{Ann}_{\mathbb{Z}[\mathrm{Gal}(K/k)]}(\underset{\substack{\text{root of unity}\\ \text{in } K}}{\mu_K}) \cdot \vartheta_{K/k} \subset \mathbb{Z}[\mathrm{Gal}(K/k)]$$

$k=\mathbb{Q}$, $K=\mathbb{Q}(\mu_N)$, $\vartheta_{\mathbb{Q}(\mu_N)/\mathbb{Q}} = \sum_{\substack{a=1\\(a,N)=1}}^{N} \left(\frac{1}{2}-\frac{a}{N}\right)\sigma_a^{-1}$

① Annihilation $\sigma_c - N_c$

Conj (Brumer)

$$\mathrm{Ann}_{\mathbb{Z}[\mathrm{Gal}(K/k)]}(\mu_K)\, \overset{\text{minus part}}{\vartheta_{K/k}^{-}} \subset \mathrm{Ann}_{\mathbb{Z}[\mathrm{Gal}(K/k)]}(\underset{\text{ideal class gp}}{Cl_K})$$

(B)

② want to know the Galois module str. of Cl_K (Cl_K^-)

(conclusion: in order to consider ① it is better not to consider ②!)

$k=\mathbb{Q}$ Stickelberger's thm (1890)

§1 $k=\mathbb{Q}$ $\mathrm{Ann}(\mu_K)\vartheta_{K/k} \subset \mathrm{Ann}(Cl_K)$

$$\bigcap \qquad \cup\!\shortmid \quad \mathrm{Fitt}_{\mathbb{Z}[\mathrm{Gal}(K/k)]}(Cl_K)$$

R : comm. ring

M : finitely presented R-module

$$R^m \xrightarrow[A]{} R^n \longrightarrow M \longrightarrow 0 \quad \text{exact}$$

$$\begin{array}{ll} \mathrm{Fitt}_R(M) = (\text{ideal gen. by } n\times n\text{-minors of } A) \\ \cap \qquad\qquad \subset R \\ \mathrm{Ann}_R(M) \end{array}$$

<u>Examples</u> $R = \mathbb{Z}$, M : finite $\mathbb{Z}$-module

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$$

one can take $m = n$.

$$\mathrm{Fitt}_{\mathbb{Z}}(M) = (\det A) = (\# M)$$

$$\mathrm{Ann}(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = (2), \quad \mathrm{Fitt}(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = (4)$$

$R = \mathbb{Z}_p[[T]]$

M: fin. gen. torsion R-module with no non-trivial finite submodule.

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

one can take $m = n$ and $\mathrm{Fitt}_R(M) = (\det A)$

$$= \mathrm{char}_R(M)$$

$k = \mathbb{Q}$, $K = \mathbb{Q}(\mu_N)$

$\Theta_{\mathbb{Q}(\mu_N)}$: Iwasawa-Sinnott's ideal

$$\Theta_{\mathbb{Q}(\mu_N)} = \Theta'_{\mathbb{Q}(\mu_N)} \cap \underbrace{\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})]}_{=R}$$

$$\Theta'_{\mathbb{Q}(\mu_N)} = \langle \{ \nu\, \vartheta_{\mathbb{Q}(\mu_d)} \mid d \mid N \} \rangle_R \subset R \otimes \mathbb{Q}$$

$$\nu : \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})] \longrightarrow \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})]$$

$$\sigma \longmapsto \sum_{\tau|_{\mathbb{Q}(\mu_d)} = \sigma} \tau$$

Theorem (with Miura)

$$\mathrm{Fitt}(Cl^{-}_{\mathbb{Q}(\mu_N)}) \otimes \mathbb{Z}[\tfrac{1}{2}] = \underbrace{\Theta_{\mathbb{Q}(\mu_N)}}_{\text{not principal}} \otimes \mathbb{Z}[\tfrac{1}{2}]$$

proof Main Conj, Wiles technique, in general

$\vartheta_{K/\mathbb{Q}}$ is explicit, $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) = I_{\ell_1} \times \cdots \times I_{\ell_r}$

$N = \ell_1^{e_1} \cdots \ell_r^{e_r}$

Cor (Mazur, Wiles, Solomon)

p: odd $\psi : (\mathbb{Z}/N\mathbb{Z})^{\times} \longrightarrow \mathbb{C}^{\times}$ odd char.

$K_\psi \longleftrightarrow \mathrm{Ker}\,\psi$

$$\#(Cl_{K_\psi} \otimes \mathbb{Z}_p)^{\psi} = \#\, \mathbb{Z}_p[\mathrm{Im}\,\psi]/(B_{1,\psi^{-1}})$$

except the case

$K_\psi = \mathbb{Q}(\mu_{p^n})$, $\psi|_{(\mathbb{Z}_p)^{\times}} = \omega$

For a general tot. real field k,

(B) $\mathrm{Ann}(\mu_{p^\infty}(K))\,\vartheta_{K/k} \subset \mathrm{Ann}(A_K)$, $A_K = Cl_K \otimes \mathbb{Z}_p$

(SB) $\mathrm{Ann}(\mu_K)\,\vartheta_{K/k} \subset \mathrm{Fitt}(Cl_K)$

does not hold in general

§2. Wiles

k: tot. real

p: odd prime

K: CM field s.t $p \nmid [K:k]$ only in this section

K/k: abelian.

$$K \subset K_n \subset K_\infty \quad (K_\infty/K\text{: cyclotomic } \mathbb{Z}_p\text{-ext})$$

$A_{K_n} = Cl_{K_n} \otimes \mathbb{Z}_p \qquad X_{K_\infty} = \varprojlim A_{K_n}$

χ: odd char s.t $\mathrm{cond}(\chi) = \mathrm{cond}(K)$

$\chi \in \mathrm{Gal}(K/k)^{\wedge}$

<u>Thm</u> (Wiles)
(Main Conj) If $\chi \neq \omega$,

$$\mathrm{char}(X_{K_\infty}^{\chi}) = (\vartheta_{K_\infty/k}^{\chi})$$

(NT2) No prime above p splits in K/K^+

$$(X_{K_\infty}^{-})_{\mathrm{Gal}(K_\infty/K_n)} \xrightarrow{\sim} A_{K_n}^{-}$$

<u>Cor</u> Assume (NT2)
$\Rightarrow \vartheta_{K_n}^{\chi} \in \mathrm{Fitt}(A_{K_n}^{\chi})$ for all $n \geq 0$.

<u>Cor</u> Assume (NT2), $p \nmid [K:k] \Rightarrow$ (B)

<u>Thm</u> (Wiles)
Assume $\underbrace{(\mathrm{NT2})}_{p \nmid [K:k]}$ or $K^{\mathrm{Gal\,cl}} \not\subset (K^{\mathrm{Gal\,cl}})^+(\mu_p)$

Galois closure (over $\mathbb{Q}$)

then (B) holds

this assumption is needed

(Gap of Wiles)
He does not assume $p \nmid [K:k]$
and claims that (B) reduces the
case $p \nmid [K:k]$

K
$G\,(\,|\,)$ p-ext
K'
$|\,)$ prime to p
k

Want $\vartheta \cdot A_K = 0$
$\not\rightarrow$ want $(\vartheta A_K)_G = 0$
(He only proved $c_{K/K'}(\vartheta)\cdot(A_K)_G = 0$
$c_{K/K'} : \mathbb{Z}_p[\mathrm{Gal}(K/k)] \longrightarrow \mathbb{Z}_p[\mathrm{Gal}(K/k)]$

§3. Tate, ⋯, Greither, Nickel

$\begin{matrix} k \\ | \\ K \end{matrix}$ K/k finite abelian. $K^{\mathrm{Gal.cl}} \not\subset (K^{\mathrm{Gal.cl}})^{+}(\mu_p)$

$\Downarrow$

<u>Thm</u> (Nickel) Assume (NT2) or (W)

- $\mu_{K_\infty/k} = 0$ ← cyclo. $\mathbb{Z}_p$
- (*) For $\forall$ prime $\mathfrak{p}$ of k above p, $\mathfrak{p}$ is tame in K or no prime above $\mathfrak{p}$ splits in K/K^+.

Then $\underbrace{\mathrm{Ann}(\mu_{p^\infty}(K))\,\vartheta_{K/k} \subset \mathrm{Fitt}(A_K)}_{(SB)}$.

In particular (B) holds.

If (*) does not hold, (SB) does not hold for K_n $n \gg 0$. → explicit counter examples in [Greither - Kurihara]

T: finite set of primes in k

A_K^T : ray class gp $\prod_{\mathfrak{p} \in T} \mathfrak{p}$

$$0 \longrightarrow R^n \xrightarrow[A]{} R^n \longrightarrow A_K^T \longrightarrow 0$$

$$\mathrm{Fitt}(A_K^T) = (\vartheta_K^T)$$

a modified Stickelberger element

$$\vartheta_K^T = \prod_{v \in T}(1 - \mathrm{Frob}_v^{-1} N v)\,\vartheta_{K/k} \qquad \mathrm{Ann}(\mu_K)\vartheta_{K/k}$$