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Second talk

§4. Greither - Popescu (today's reference [5])

k : totally real

K : CM-field s.t K/k finite abel.

$S \supset S_{ram}(K/k) \cup S_\infty$

p : odd , T : finite set of primes of k

s.t $T \cap S = \phi$, $T \neq \phi$

$\{\zeta \in \mu_K \mid \zeta \equiv 1 \mod \prod_{v \in T} v\} = \{1\}$

Assume $\mu_{K_\infty/K}$ $\quad K_\infty/K$ cyclo. $\mathbb{Z}_p$-ext.

Construct $J = T_p(M_{S,T}^{K_\infty})^-$: $\Lambda_{K_\infty} = \mathbb{Z}_p[[Gal(K_\infty/k)]]$-module

Th (1) $pd_{\Lambda_{K_\infty}}(J) \leq 1$

$$\left(\begin{array}{l} 0 \longrightarrow \Lambda_{K_\infty}^n \xrightarrow{A} \Lambda_{K_\infty}^n \longrightarrow J \longrightarrow 0 \\ Fitt_{\Lambda_{K_\infty}}(J) = (\det A) \end{array} \right)$$

Assume

$S \supset S_{ram}(K_\infty/k)$

$\Downarrow$

$S_{ram}(K/k) \cup S_p$

primes above S

(2) $Fitt_{\Lambda_{K_\infty}^-}(J) = (\vartheta^T_{K_\infty/k, S})$

$$\vartheta^T_{K/k,S} = \prod_{v \in T}(1 - Frob_v^{-1}(Nv)) \times \prod_{v \in S \setminus S_{ram}(K/k) \cup S_\infty} (1 - Frob_v^{-1}) \times \vartheta_{K/k}$$

(3) $J \longrightarrow Hom(\mathfrak{X}_S^+, \mathbb{Z}_p(1))$

$\mathfrak{X}_S$: Gal gp of K_∞ max. abel. pro-p unram. outside p.

$A_K^T \qquad char(\mathfrak{X}_{S_p}^+) = (p\text{-adic } L)$

$$\mathfrak{X}_S \twoheadrightarrow \mathfrak{X}_{S_p}$$

$\mathfrak{X}_{S_p}$ free $\mathbb{Z}_p$-module of finite rk

$$\vartheta^T_{K_\infty/k,S} \cdot \mathcal{J} = 0 \Rightarrow (\vartheta^T_{K_\infty/k,S})^\sim \cdot \mathfrak{X}^+_{S_p} = 0$$

$$\mathfrak{X}^+_{S_p} = \mathrm{Hom}(\underbrace{\varinjlim A^-_{K_n}}_{\text{Kummer theory}\ \| \ A^-_{K_\infty}}, \mathbb{Q}_p/\mathbb{Z}_p(1))$$

$$(\vartheta^T_{K_\infty/k}) \cdot A^-_{K_\infty} = 0$$

$$\vartheta^T_{K_\infty/k,S} \cdot A^-_K = 0$$

$$\vartheta^T_{K/k,S} \cdot A^-_K = 0$$

Assume $\mu_{K_\infty/K} = 0$

Cor $\mathrm{Ann}(\mu_{p^\infty}(K)) \cdot \vartheta^-_{K/k,S_p} \subset \mathrm{Ann}(A^-_K)$

$$S = S_{ram}(K/k) \cup S_p$$

Cor Assume $\mu_{K_\infty/K} = 0$ and any prime $\mathfrak{p}$ above p

is ramified in k

or

does not split in K/K^+.

then (B) holds $\mathrm{Ann}(\mu_{p^\infty}(K))\, \vartheta^-_{K/k} \subset \mathrm{Ann}(A^-_K)$

§ 5. Non commutative Fitting invariant

G : finite (Nickel [9], [6])

$R = \mathbb{Z}_p[G]$

$$A = R \otimes \mathbb{Q}_p = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_t}(D_t)$$ ③

D_i: skew field.

$\zeta(D_i) = F_i \qquad \dim_{F_i} D_i = s_i^2$

$\zeta(A) = F_1 \oplus \cdots \oplus F_t$

reduced norm

$nr : A \longrightarrow \zeta(A) = F_1 \oplus \cdots \oplus F_t$

componentwise

$$\begin{array}{ccccc} M_{n_i}(D_i) & \longrightarrow & M_{n_i s_i}(E_i) & \xrightarrow{\det} & E_i \\ & & & & \cup \\ M_{n_i}(D_i) \otimes E_i & = & M_{n_i s_i}(E_i) & & F_i \end{array}$$

$[E_i : F_i] < \infty$

<u>Ex</u> $\mathbb{H} = \{a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}\}$

$\mathbb{H} \longrightarrow M_2(\mathbb{C}) \quad a+bi+cj+dk$

$$\mapsto \begin{pmatrix} a+bi & -c-di \\ c-di & a-bi \end{pmatrix}$$

$$\underset{\det}{\mapsto} a^2+b^2+c^2+d^2$$

$nr(R) \not\subset R$ in general

<u>Ex</u> $G = D_{2p} = \langle \sigma, \tau \rangle$

$\sigma^p = \tau^2 = 1 \qquad \tau\sigma = \sigma^{-1}\tau$

$$nr(\sigma+\tau) = \frac{1}{p} \sum_{\alpha \in D_{2p}} \alpha \notin R$$

R': maximal $\mathbb{Z}_p$-order in A.

$$nr(R) \subset R' \subset \zeta(R') \qquad \zeta(A)$$

$$\zeta(R') = R'_1 \oplus \cdots \oplus R'_t \qquad \cup$$

$$I = \langle nr(H) \mid H \in M_b(R), b \in \mathbb{Z}_{>0} \rangle_{\zeta(R)} \subset \zeta(R')$$

$\mathbb{Z}_p$-order

Def M : finitely presented left R-module

$$\begin{array}{ccc} R^m & \xrightarrow{h} & R^n \\ x & \longmapsto & xA \end{array} \quad \text{presentation}$$

$S_n(A)$: set of $n\times n$ submatrixes of A

$$\mathrm{Fitt}_R(h) = \begin{cases} 0 & \text{if } m<n \\ \langle \mathrm{nr}(H) \mid H \in S_n(A) \rangle_I \end{cases}$$

I-module $\subset \mathfrak{Z}(R')$

This <u>does</u> depend on the choice of h

Prop If $m=n$, $\mathrm{Fitt}(h)$ <u>does not</u> depend on h.

In this case, we define

$$\mathrm{Fitt}_R(M) = \mathrm{Fitt}_R(h).$$

In general,

$\exists!$ maximal $\mathrm{Fitt}(h)$

define $\mathrm{Fitt}_R^{\max}(M)$ by this maximal one.

G : abelian $\Rightarrow$ $\mathrm{Fitt}_R(M) \subset \mathrm{Ann}_R(M)$

(not true in non comm. cases!)

Lemma For $H \in M_b(R)$

(there exists) $H^* \in M_b(R')$ s.t.

$HH^* = H^*H = \mathrm{nr}(H)\,\mathrm{id}.$ $\lrcorner$

Def $\mathcal{H} = \{ x \in \mathfrak{Z}(R) \mid xH^* \in M_b(R) \text{ for all } H \in M_b(R) \text{ and all } b \in \mathbb{Z}_{>0} \}$

$\mathcal{H}$ is an ideal of I.

Prop $\mathcal{H}\cdot \mathrm{Fitt}_R^{max}(M) \subset \mathrm{Ann}_R(M)$

(pf) $R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$

$x \longmapsto xA$

$H \in M_n(R)$ $\quad x \in \mathcal{H}$

$$\begin{tikzcd}
R^n \arrow[r, "H"] \arrow[dr, "xH^*"'] & R^n \arrow[r] \arrow[d, "x\,\mathrm{nr}(H)"] & \mathrm{Coker}(H) \arrow[r] \arrow[d, "x\,\mathrm{nr}(H) = 0"] & 0 \\
R^n \arrow[r, "H"'] & R^n \arrow[r] & \mathrm{Coker}(H) \arrow[r] & 0
\end{tikzcd}$$

If G abelian, $\mathcal{H} = R$.

If $R = R'$, $\mathcal{H} = \mathfrak{Z}(R) = \mathfrak{Z}(R')$

Prop (Nickel, Johnston)

G' : comm. subgp of G.

If $p \nmid \# G'$, $\mathcal{H} = \mathfrak{Z}(R) = I$.

Example $\mathbb{Z}_2[D_{2p}]$

§6. Non comm. Brumer conj

k : tot. real

K : CM field s.t K/k finite Galois, $G = \mathrm{Gal}(K/k)$

p : prime

$R = \mathbb{Z}_p[G]$

S, T

$S \supset S_{ram}(K/k) \cup S_\infty$

$T \cap S = \phi$, $T \neq \phi$. $\{\zeta \in \mu_K \mid \zeta \equiv 1 \mod \prod_{v \in T} v\} = \{1\}$

$$\vartheta_{K/k,S} = \sum_{\chi \in \mathrm{Irr}\, G} L_S(0, \chi^{-1}) e_\chi$$

$$e_\chi = \frac{\chi(1)}{\# G} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$$

$$\delta_T(\chi) = \prod_{v \in T} \det(1 - (Nv)\mathrm{Frob}_v^{-1} \mid V_\chi)$$

$$\delta_T = \sum_{\chi} \delta_T(\chi) e_\chi$$

$$\mathfrak{A}_S = \langle \delta_T \mid T \text{ satisfies above} \rangle_{\zeta(R)} \subset \zeta(R')$$

<u>Conj</u> (B) (Non comm. Brumer)

$$\mathcal{H} \cdot \mathfrak{A}_S \cdot \vartheta_{K/k,S} \subset \mathrm{Ann}(A_K)$$

<u>Theorem</u> (Nickel [14])

Assume $\mu_{K_\infty} = 0$ and $S \supset S_p$

$$\mathcal{H} \cdot \mathfrak{A}_S \cdot \vartheta_{K/k,S} \subset \mathrm{Ann}(A_K)$$

<u>Cor</u> Assume $\mu_{K_\infty/K} = 0$

$\forall \mathfrak{p}$: prime of k above p

$\mathfrak{p}$ ramified in K

or

$\mathfrak{p}$ does not split in K/K^+

Then (B) holds.