Francesco Lemma

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Review of a paper by Piatetski-Shapiro

L-function for $GSp(4)$

(after Piatetski-Shapiro Pacific Jour. of Math)

0. Introduction
1. $GSp(4)$ and its subgroups
2. Eisenstein series and global L-functions

§0. (V, ψ) 4-dim. symplectic space $/\mathbb{Q}$

ψ non deg. alternate form on V.

$$GSp(4)_{/\mathbb{Q}} = \{ g \in GL(V) \mid \psi(gv, gw) = \nu(g)\psi(v,w),\ \nu(g) \in \mathbb{G}_m \ \forall v, w \in V \}$$

π : auto. cusp. irred. rep of $GSp(4)$.

$${}^{L}GSp(4)^{\circ} \cong GSp(4)(\mathbb{C})$$

$$r : GSp(4)(\mathbb{C}) \hookrightarrow GL(4)(\mathbb{C})$$

$S = \{\infty\} \cup \{\text{ramified places of } \pi\}$ (finite)

$$L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r)$$: spinor (degree 4)

L-function of π absolutely convergent

for $\mathrm{Re}(s) \gg 0$

Goals:

- provide an integral rep. of $L_S(s, \Pi)$

Important in arith. applications

- Harris "Occult period invariant ..."
- Lemma "Higher regulators, ..."

- provide a definition of $L(s, \Pi_v)$ for $v \in S$
- show analytic continuation and functional equation

Main tool: generalized Whittaker (Bessel) model of Π.

§1. $GSp(4)$ and its subgroup.

k: field of char $\neq 2$, $GSp(4)/_k$

C: center , P: Siegel parabolic

$$P = \left\{ \left(\begin{array}{c|c} \alpha A & AM \\ \hline 0 & {}^tA^{-1} \end{array} \right) \middle| \ A \in GL(2), \alpha \in \mathbb{G}_m, M \in M_2, {}^tM = M \right\}$$

$$P = (\mathbb{G}_m \times GL_2) \ltimes S \text{ where } S = \left\{ \left(\begin{array}{c|c} I_2 & M \\ \hline & I_2 \end{array} \right) \ {}^tM = M \right\}$$

any hom $S \to k$ is of the form $A \to tr(\beta A)$ for $\beta \in M_2(k)$ ${}^t\beta = \beta$. We call it non-deg. if $\det \beta \neq 0$

Fix a non-deg. hom. $l_\beta : S \to k$

Let D be the conn. component of the stabilizer of l_β in $GL_2 \times \mathbb{G}_m$

<u>Lemma</u> There exists 2-dim k-alg K_β s.t $D = \mathrm{Res}_{K_\beta/k} \mathbb{G}_m$. Furthermore, K_β is

$$\begin{cases} \text{a quad. ext. of } k & \text{if } \beta \text{ is anisotropic} \\ k \times k & \text{else} \end{cases}$$

$K = K_\beta$

$N = \{ s \in S \mid l(s) \, (= \mathrm{tr}(\beta s)) = 0 \}$

$G = \{ g \in GL_2(K) \mid \det g \in k^\times \}$

$V = K^2 \curvearrowleft G$

symplectic form on V: $\rho(x, y) = \mathrm{tr}_{K/k}(x_1 y_2 - x_2 y_1)$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

V: 4-dim k-vect. space with symplectic form

we have $G \hookrightarrow GSp_\rho = \{ g \in GL_4(k) \mid \rho(gx, gy) = \lambda \rho(x, y) \}$

<u>Notation</u> $x \mapsto \bar{x}$ is the non-trivial k-involution of K.

<u>Proposition</u> Fix $\Psi \simeq \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ (beginning)

Then there exists an isom. $GSp_\rho \simeq GSp_4$ s.t $G \cap R = D \ltimes N$ where $R = D \ltimes S$

2. Eisenstein series and global L-functions

k: global field, ψ non deg characters on $S_{\mathbb{A}}$.

ν character on $D_{\mathbb{A}} = I_K$ ideles of $K = \mathbb{A}_K^\times$

$$\left(\psi(s) = \Psi(\mathrm{tr}(\beta s)) \cdot \det\beta \neq 0\right)$$

$V_{\mathbb{A}} \curvearrowleft G_{\mathbb{A}}$

__Def__ $S(V_{\mathbb{A}})$ the space of Schwartz-Bruhat funct. on $V_{\mathbb{A}}$ is the space of finite linear combinations of $\prod_v \phi_v$ where ϕ_∞ is in the usual Schwartz space and ϕ_v for $v < \infty$ is loc. const. with compact support and $1_{\mathcal{O}_{k,v}}$ for almost all v.

$\phi \in S(V_{\mathbb{A}})$ $\mu : k^\times \backslash I_k \to \mathbb{C}^\times$

$$f^\phi(g, \mu, \nu, s) = \mu(\det g)|\det g|^{s+\frac{1}{2}} \int_{I_K} \phi((0,t)g)|t\bar{t}|^{s+\frac{1}{2}} \times \mu(t\bar{t})\nu(t)\,d^\times t$$

function on $G_{\mathbb{A}}$.

$|\cdot|$ usual idele norm.

$B'_{\mathbb{A}}$ the Borel of $G_{\mathbb{A}}$

$$\chi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{t} & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) = \mu(x)|x|^{s+\frac{1}{2}}\nu^{-1}(t)$$

well defined char on $B'_{\mathbb{A}}$ then $f^\phi(g, \mu, \nu, s) \in \mathrm{ind}_{B'_{\mathbb{A}}}^{G_{\mathbb{A}}} \chi$

Eisenstein series $E^{\phi}(g,\mu,\nu,s) = \sum_{\gamma \in B_k \backslash G_k} f^{\phi}(\gamma g,\mu,\nu,s)$ ⑤

<u>Theorem</u> (Jacquet, Autom form on GL(2) II)

$E^{\phi}(g,\mu,\nu,s)$ is a holomorphic functions of $s \in \mathbb{C}$ except for a finite number of $\underbrace{\text{poles}}_{\text{at most } 2}$ and satisfies

a funct. eq

$$E^{\phi}(g,\mu,\nu,s) = E^{\hat{\phi}}(g,\mu^{-1}\nu^{-1},\bar{\nu},1-s)$$

where $\nu^{-1} = \nu|_{I_k}$, $\bar{\nu}(a) = \nu(\bar{a})$, $\hat{\phi}$ is the Fourier transform.

(Π, V_Π) be a automorphic cuspidal rep of $GSp(4)/k$

$\exists(\nu,\psi)$ as above $\exists \varphi \in V_\Pi$ s.t. $\int_{C_A R_k \backslash R_A} \varphi(r)\alpha_{\nu,\psi}(r)^{-1}dr \neq 0$

(Howe) $R = D \ltimes S$ and

$$\alpha_{\nu\psi}(ds) = \nu(d)\psi(s)$$

Generalized Whittaker functionnal (a Bessel functionnal)

$W_\varphi : GSp(4,\mathbb{A}) \longrightarrow \mathbb{C}$ satisfies $W_\varphi(rg) = \alpha_{r\psi}(r)W_\varphi(g)$ $\forall r \in R$

$$g \longmapsto \int_{C_A R_k \backslash R_A} \varphi(rg)\alpha_{\nu,\psi}^{-1}(r)\,dr$$

Prop

$$Z = \int_{C_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \varphi(g) E^{\phi}(g,\mu,\nu,s)\, dg$$

$$= \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} W_{\varphi}(g) f^{\phi}(g,\mu,\nu,s)$$

Rem In the right integral, there are no more k-points

Sketch of proof

$$Z = \int_{C_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \sum_{\gamma \in B'_k \backslash G_k} \varphi(\gamma g) f^{\phi}(\gamma g,\mu,\nu,s)$$

$$\begin{array}{c} B_k \backslash G_{\mathbb{A}} \\ | \\ C_{\mathbb{A}} B_k \backslash G_{\mathbb{A}} \\ | \\ C_{\mathbb{A}} G_k \backslash G_{\mathbb{A}} \end{array}$$

$$Z = \int_{C_{\mathbb{A}} B'_k \backslash G_{\mathbb{A}}} \varphi(g) f^{\phi}(g,\mu,\nu,s)\, dg$$

Fourier expansion : $\varphi(g) = \sum_{\psi \in \mathrm{Char}(S_k \backslash S_{\mathbb{A}})} \varphi_{\psi}(g)$

where $\varphi_{\psi}(g) = \int_{S_k \backslash S_{\mathbb{A}}} \varphi(sg) \psi^{-1}(s)\, ds$

left invariance of f^{ϕ} under $N_{\mathbb{A}}$

$\Omega = \{$ non-trivial char $S_k \backslash S_{\mathbb{A}}$ trivial on $N_{\mathbb{A}} \}$

$$Z = \int_{C_{\mathbb{A}} H_k D_k N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \sum_{\psi \in \Omega} \varphi_\psi(g) f^\phi(g, \mu, \nu, s)\, dg$$

with decomposition $B' = HDN$

$H_k \curvearrowright \Omega$ simply transitively

$$Z = \int_{C_{\mathbb{A}} H_k \cdot D_k N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \sum_{h \in H_k} \varphi_\psi(hg) f^\phi(hg, \mu, \nu, s)\, dg$$

$$= \int_{C_{\mathbb{A}} D_k N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi_\psi(g) f^\phi(tg; \mu, \nu, s)\, dg$$

$$= \cdots = \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \int_{C_{\mathbb{A}} D_k \backslash N_k} \varphi_\psi(tg) \nu^{-1}(t)\, dt\, f^\phi dg$$

def Fourier coeff

$$\doteq \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \int_{C_{\mathbb{A}} D_k \backslash D_{\mathbb{A}}} \int_{S_k \backslash S_{\mathbb{A}}} \varphi(tsg) \psi^{-1}(s) \nu^{-1}(t)\, dt\, ds\, f^\phi dg$$

$$= \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} W_\varphi(g) f^\circ(g, \mu, \nu, s)\, dg$$

Prop. $\Rightarrow Z = \prod_v Z_v$

<u>Prop.</u> (Andrianov
Furusawa unpublished)

v unramified $\phi_v = 1_{O_{k_v}}$.

ϕ invariant under $GSp(4, \mathcal{O}_{k_v})$

then $Z_v = L(s, \Pi_v)$

analytic continuation and

functional eq follows from the ones of

$E^{\phi}(g, \mu, \nu, s)$ (roughly)

$$\frac{\int W_\varphi(g) f^{\phi}(-)\, dg}{L(s, \pi_v)}$$ is holomorphic