2012/4/4 talk III

To prove Iwasawa Main Conj., we generalize the method of Ribet

- from mod p rep $\rightsquigarrow$ mod p^n, $\mathbb{Z}_p$, $\mathbb{Q}_p/\mathbb{Z}_p$-rep
- congruence between a cuspform and Eisenstein series
 $\rightsquigarrow$ congruence between a p-adic family of cuspforms and a p-adic family of Eisenstein series

For simplicity, we assume $\psi \neq \mathbb{1}$.

By Analytic Class number formula, it suffices to show that

(☆) For any ht one prime $\mathfrak{p}$ of Λ_ψ

$$\mathrm{ord}_{\mathfrak{p}}(L_p(\psi)) \leq \mathrm{ord}_{\mathfrak{p}}\left(\mathrm{char}_{\Lambda_\psi}(X_{K_{\omega\psi^{-1}}})_{\omega\psi^{-1}}\right)$$

<u>Part (A)</u>

Let $N \in \mathbb{N}$ s.t $(N,p)=1$

By a well-known description of q-expansion of $E_{k,\eta}$, we have

$$\mathbb{E}_\psi = \sum_{n=0}^{\infty} A_n(\mathbb{E}_\psi) q^n \in \Lambda_\psi[[q]]$$

for any even Dirichlet char ψ of conductor N or Np

where $A_n(\mathbb{E}_\psi) = \begin{cases} \mathrm{Tw}_{K_{cyc}}\left(L_p(\psi\omega^2)^{\iota}/2\right) \\ \sum_{0<d\mid n} \psi(d)\, d \cdot \gamma^{s(d)} \end{cases}$

$(\chi_{cyc}(\gamma))^{s(d)} := \langle d \rangle$

For any $\forall \chi_{cyc}^{k-2}\phi : \Gamma_{cyc} \to \overline{\mathbb{Q}}_p^\times$ ($k \geq 2$, ϕ is finite)

$(\chi_{cyc}^{k-2}\phi)(\mathbb{E}_\psi) \in \mathbb{Z}_p[\phi][[q]]$

is (the p-stabilization of) $E_{k,\psi\omega^{2-k}\phi}$.

Def. Let $\mathbb{I}$ be a domain finite flat over Λ_{cyc}.
A formal q-expansion $\mathbb{F} = \sum_{n=0}^{\infty} A_n(\mathbb{F}) q^n \in \mathbb{I}[[q]]$
is called an $\mathbb{I}$-adic modular form of level Np^∞
of Neben char. ψ if

$\kappa(\mathbb{F}) \in \kappa(\mathbb{I})[[q]]$ is a classical form of
level Np^*, wt k with Neben char. $\psi\phi\omega^{2-k}$
for any ring hom. $\kappa : \mathbb{I} \to \overline{\mathbb{Q}}_p$ s.t. $\kappa|_{\Lambda_{cyc}} = \chi_{cyc}^{k-2}\phi$
($\exists k, \phi$).

Wiles' idea ① Consider $\mathbb{I}$-adic families and
localize them at ρ.
Wiles' idea ② Avoid the construction of $\mathbb{I}$-adic
forms with unit constant term.

For any finite order char. ρ of Γ_{cyc}, we define

$\mathbb{G}_\rho = E_{1,\rho^{-1}\omega^{-1}} \cdot Tw_{\chi_{cyc}\rho}(\mathbb{E}_\psi)$

$\mathbb{G}_\rho$ is a Λ_ψ-adic modular form with Neben char. ψ.
If the order of ρ is large enough, $A_0(\mathbb{G}_\rho)$ is prime to $A_0(\mathbb{E}_\psi)$.

Put $\mathbb{F}' = A_0(\mathbb{G}_\rho)\mathbb{E}_\psi - A_0(\mathbb{E}_\psi)\mathbb{G}_\rho$.

By construction,

- $\mathbb{F}'$ is a semi-cusp form
- $\mathbb{F}' \equiv \mathbb{E}_\psi \mod \mathfrak{p}^\ell$ (modulo mult. by a unit of $(\Lambda_\psi)_{\mathfrak{p}}$.)

Wiles' idea ③

Construct cuspform directly from a non-eigen semi-cuspform

Put $\mathbb{F}'' = \lim_{n\to\infty} T_p^{n!} \cdot T_N \cdot \prod_{\ell \mid N} \left(T_\ell^{\phi(Np)} - (\ell \gamma^{s(\ell)})^{\phi(Np)}\right) \mathbb{F}'$

Then, $\mathbb{F}''$ is a cuspform (cf. Wiles 1990, Lemma 3.2).
(but eigenform only mod $\mathfrak{p}^\ell$)

$\Downarrow$

$\exists\, \mathbb{I}$: finit flat ext. of Λ_ψ, $\tilde{\mathfrak{p}}$: prime of $\mathbb{I}$ over $\mathfrak{p}$
$\mathbb{F} \in \mathbb{I}[[q]]$ s.t

- $\mathbb{F}$ is eigen cuspform
- $\mathbb{F} \equiv \mathbb{E}_\psi \mod \tilde{\mathfrak{p}}^\ell$

Part (B)

Thanks to Hida, we have:

Thm. $\exists!\ \rho_{\mathbb{F}} : G_{\mathbb{Q}} \xrightarrow[\text{cont.}]{} \mathrm{Aut}(V_{\mathbb{F}}) \cong GL_2(\mathbb{K}^2)$

where $\mathbb{K} = \mathrm{Frac}(\mathbb{I})$

s.t. $\cdot$ $\rho_{\mathbb{F}}$ is irred. and unr. outside Np.

$\cdot$ $\mathrm{tr}(\rho_{\mathbb{F}}(\mathrm{Frob}_\ell^{\text{arith}})) = A_\ell(\mathbb{F})$ for $\forall \ell \nmid Np$

$\cdot$ $\det \rho_{\mathbb{F}} \cong \mathbb{K}(\hat{\chi}\chi_{cyc}\omega\psi_{\mathbb{F}})$

where $\hat{\chi} : G_{\mathbb{Q}} \twoheadrightarrow \Gamma_{cyc} \hookrightarrow \Lambda^\times \hookrightarrow \mathbb{I}^\times$ $\lrcorner$

Since $\rho_{\mathbb{F}}$ is continuous, for each ht one prime $\tilde{\mathfrak{p}}$ of $\mathbb{I}$, we have an $\mathbb{I}_{\tilde{\mathfrak{p}}}$-lattice

$$\mathbb{T} \cong \mathbb{I}_{\tilde{\mathfrak{p}}}^{\oplus 2} \subset V_{\mathbb{F}} \curvearrowleft G_{\mathbb{Q}}.$$

Further, since $\mathbb{F} \equiv \mathbb{E}_\psi \mod \tilde{\mathfrak{p}}$,

$$\left(\mathbb{T}/\tilde{\mathfrak{p}}\mathbb{T}\right)^{s\text{-}s} \cong \left(\mathbb{I}_{\tilde{\mathfrak{p}}}/\tilde{\mathfrak{p}}\right)(\mathbb{1}) \oplus \left(\mathbb{I}_{\tilde{\mathfrak{p}}}/\tilde{\mathfrak{p}}\right)\overset{\tilde{\eta}}{\overset{\shortparallel}{(\tilde{\chi}\chi_{cyc}\omega\psi_{\mathbb{F}})}}$$

Note that

any simple subquotient of any $\mathbb{I}_{\tilde{\mathfrak{p}}}$-lattice $\mathbb{T}$ is either $(\mathbb{I}_{\tilde{\mathfrak{p}}}/\tilde{\mathfrak{p}})(\mathbb{1})$ or $(\mathbb{I}_{\tilde{\mathfrak{p}}}/\tilde{\mathfrak{p}})(\tilde{\eta})$.

We call the former (resp. the latter) type $\mathbb{1}$ (resp. type $\tilde{\eta}$).

We prove that there exists an $\mathbb{I}_{\hat{\mathfrak{P}}}$-lattice $\mathbb{T} \subset V_{\mathbb{F}}$ s.t

(1) $0 \to A \to \mathbb{T}/\hat{\mathfrak{P}}^{\ell}\mathbb{T} \longrightarrow B \longrightarrow 0$

with $A \simeq B \simeq \mathbb{I}_{\hat{\mathfrak{P}}}/\hat{\mathfrak{P}}^{\ell}$ as $\mathbb{I}_{\hat{\mathfrak{P}}}$-module every simple subquot. of A (resp. B) is type $\mathbb{1}$ (resp. type $\hat{\eta}$).

(2) $\mathbb{T}/\hat{\mathfrak{P}}^{\ell}\mathbb{T}$ has no type $\mathbb{1}$-quotient.

We only note that (2) is a conseq of the irreducibility of $G_{\mathbb{Q}} \curvearrowright V_{\mathbb{F}}$.

Recall the following local description:

<u>Thm (Mazur-Wiles, Wiles)</u>

Let $\rho_{\mathbb{F}} : G_{\mathbb{Q}} \to \mathrm{Aut}(V_{\mathbb{F}})$ restricted to $G_{\mathbb{Q}_p}$ has filtration

$$0 \longrightarrow \mathbb{K}(\hat{\alpha}) \longrightarrow V_{\mathbb{F}} \longrightarrow \mathbb{K}(\hat{\kappa}\chi_{cyc}\omega\psi_{\mathbb{F}}\hat{\alpha}^{-1}) \longrightarrow 0$$

$G_{\mathbb{Q}_p}$ acting on $V_{\mathbb{F}}$,

where $\hat{\alpha}$ is an unr. char : $G_{\mathbb{Q}_p} \longrightarrow \mathbb{I}^{\times}$ s.t

$\hat{\alpha}(\mathrm{Frob}_p^{\mathrm{arith}}) = A_p(\mathbb{F})$ $\lrcorner$

<u>Rmk</u> This is proved in Wiles 1986 by "arith geom." of $J_1(Np^n)$ and Local Langlands correspondence $\lrcorner$

As in the case of Ribet's proof (the second talk), we have an element

$$C \in H^1(\mathbb{Q}, (\mathbb{I}_{\tilde{\wp}}/\tilde{\wp}^{\ell})(\tilde{\kappa}^{-1}\kappa_{cyc}^{-1}\omega^{-1}\psi_{\mathbb{F}}^{-1}))$$

s.t

- $\tilde{\wp}^{\ell-1} \cdot C \neq 0$
- C is locally trivial at every primes.

Since $\tilde{\kappa}^{-1}\kappa_{cyc}^{-1}\omega^{-1}\psi_{\mathbb{F}}^{-1}$ has values in $\Lambda_{cyc} \otimes \mathcal{O} \subset \mathbb{I}$, We have such an element in

$$H^1(\mathbb{Q}, (\Lambda_{cyc} \otimes \mathcal{O}/\wp^{\ell})(\tilde{\kappa}^{-1}\kappa_{cyc}^{-1}\omega^{-1}\psi_{\mathbb{F}}^{-1}))$$

By Shapiro's lemma of Galois cohomology:

$$\text{“} H^1(K, M \otimes \mathbb{Z}[Gal(L/K)]) \simeq H^1(L, M) \text{”}$$

the last group is isom. to

$$H^1(K^{cyc}_{\omega\psi^{-1},\infty}, \mathcal{O}(\chi_{cyc}^{-1}\omega\psi^{-1}))$$

$$= Hom(G_{K^{\infty}_{cyc,\omega\psi^{-1}}}, \mathcal{O}(\chi_{cyc}^{-1}\omega\psi^{-1}))$$

The subgroup of the above group which consists of locally trivial cocycles is the dual of $((X_{K_{\omega\psi^{-1}}})_{\omega\psi^{-1}})^{\bullet}$