

First talk

Proof of IMC for abelian fields by using
Euler system method I

§1. Introduction

Fix p : odd prime number

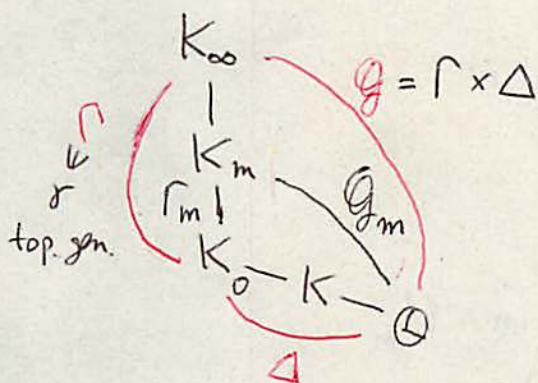
$$\begin{array}{ccc} \overline{\mathbb{Q}} & \hookrightarrow & \mathbb{C} \\ & \searrow & \text{fixed} \\ & & \overline{\mathbb{Q}}_p \end{array}$$

We write $\zeta_n = e^{2\pi i/n} \in \mu_n(\overline{\mathbb{Q}}) =: \mu_n$

$K/\mathbb{Q}$: fin. abel. ext. unr. at p

$K_m = K(\mu_{p^{m+1}})$ for $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$

($K_\infty = K_{0,\infty}^{\text{cyc}}$ in Ochiai's talk)



Iwasawa alg

$$\Lambda := \mathbb{Z}_p[[\Gamma]]$$

(caution:
notation is not the
same as the
previous talk
by Ochiai)

For each $\chi \in \hat{\Delta} := \text{Hom}(\Delta, \overline{\mathbb{Q}}_p^\times)$

$$\Lambda_\chi := \mathbb{Z}_p[[\Gamma]] \rtimes^\chi \Delta$$

M : Λ -module, $\chi \in \hat{\Delta} \rightsquigarrow M_\chi := M \otimes_\Lambda \Lambda_\chi$ (χ -quotient)

Alg. side

For a number field F ,

$$A_F := \underbrace{\text{Cl}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p}_{\text{ideal class gp of } F}$$

$$X_K := \varprojlim_m A_{K_m} \quad (\simeq \text{Ochiai's } X_{K_0} \text{ by CFT})$$

Analytic side - circular units -

(2)

$$1 - \zeta_n \in \begin{cases} \mathcal{O}_{\mathbb{Q}(\mu_n)}[\frac{1}{\ell}]^\times & \text{if } n = \ell^\nu \quad \ell: \text{prime number} \\ \mathcal{O}_{\mathbb{Q}(\mu_n)}^\times & \text{otherwise} \end{cases}$$

Def (Sinnott's circular units)

For $F/\mathbb{Q}$: a finite abel. ext.

$D(F)$: the $\mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$ -submodule of $F^\times$
generated by

$$\left\{ \pm N_{\mathcal{O}(\mu_n)/F \cap \mathcal{O}(\mu_n)}(1 - \zeta_n) \mid n \in \mathbb{Z}_{>1} \right\}$$

$$C(F) := D(F) \cap \mathcal{O}_F^\times \quad (\text{the group of circular units})$$

$\uparrow$
finite index in $\mathcal{O}_F^\times$

unit groups K_m as above

$$\bullet U_m := (\mathcal{O}_{K_m} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$$

V_1

E_m the top. closure of $\mathcal{O}_{K_m}^\times$ in U_m

(resp. $\mathcal{E}_m$)

(resp. $C(K_m)$)

$\bullet U_m$: the pro- p part of U_m

(resp. E_m, C_m)

(resp. $\mathcal{E}_m, \mathcal{C}_m$)

Fact (Leopoldt's conj for abelian fields)

$$\mathcal{O}_{K_m}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \supsetneq E_m \quad (\text{Brumer})$$

Fact (Sinnott)

$K_m^+ \subset K_m$: the max. tot. real subfield

$$\# A_{K_m^+} \sim (E_m : C_m) \quad \text{as } m \rightarrow \infty$$

We define $U_\infty := \varprojlim_m U_m$
norm

(resp. E_∞, C_∞) (resp. E_m, C_m)

Facts (from Coleman theory)

Let $\psi \in \hat{\Delta}$ be an even character.

Then ① $U_\infty, \psi, E_\infty, \psi, C_\infty, \psi$ are fin. gen. Λ_ψ -module
of rank one

$$\textcircled{2} \text{char}_{\Lambda_\psi}((U_\infty/C_\infty)_\psi) = \begin{cases} L_p(\psi) \Lambda_\psi & \text{if } \psi \neq 1 \\ \Lambda_\psi & \text{if } \psi = 1 \end{cases}$$

Here $\Lambda_\psi \rightarrow \Lambda_\psi, r \mapsto \tilde{r} = (1+p)r^{-1}$

Rmk Takae Tsuji determined Λ_ψ -module str.
of $(U_\infty/C_\infty)_\psi$ completely.

The goal of this talk

Thm (IMC)

Let $\psi \in \hat{\Delta}$: even. Then, we have

$$\textcircled{1} \text{char}_{\Lambda_\psi}(X_{K,\psi}) = \text{char}_{\Lambda_\psi}((E_\infty/C_\infty)_\psi)$$

$$\textcircled{2} \text{char}_{\Lambda_{w\psi^{-1}}}(X_{K,w\psi^{-1}}) = \begin{cases} L_p(\psi) & \text{if } \psi \neq 1 \\ \Lambda_\psi & \end{cases}$$

Fact (Ferrero - Washington)

(4)

μ -invariants of X_K and $\Lambda_\psi / (L_p(\psi))$ are zero

!!
 exponent of π in char. ideal $\leftarrow$ unit of $\mathbb{Z}_p[\psi]$

Rem $\text{char}_{\Lambda_1}(X_{K,1}) = \Lambda_1$ is "well-known"

$\mu=0 \leadsto$ reduced to $\mathbb{Q}_\infty^{\text{cyc}}/\mathbb{Q}$

So we may assume $\psi \neq 1$,

Rem By CFT

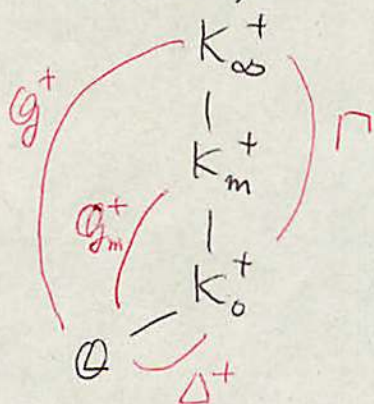
$$0 \rightarrow E_\infty/C_\infty \rightarrow U_\infty/C_\infty \rightarrow X_{K_0} \rightarrow X_K \rightarrow 0$$

$\mu=0 \leadsto$ all terms in exact seq are fin. gen. $\mathbb{Z}_p$ -module.

$$\frac{\text{char}((E_\infty/C_\infty)_\psi)}{\text{char}(X_{K,\psi})} = \frac{\text{char}((U_\infty/C_\infty)_\psi)}{\text{char}(X_{K,\psi})}$$

$\textcircled{1} \Leftrightarrow \textcircled{2}$

Strategy



We put

$X_K^+, E_\infty^+, C_\infty^+$ etc. for $K_\infty^+/\mathbb{Q}$

Note For even $\psi \in \hat{\Delta}$

$$(X_K^+)_\psi \xrightarrow[p \neq 2]{\sim} (X_K)_\psi \quad (E_\infty^+/C_\infty^+)_\psi \xrightarrow{\sim} (E_\infty/C_\infty)_\psi$$

$\Delta^+ \rightarrow \Delta^+ \rightarrow \Delta^+ \rightarrow \dots$

ker and cokernel are annihilated by 2.

We use

I. $\mu = 0$ for $(X_K)_\psi$ and $(E_\infty/C_\infty)_\psi$

II. $\begin{cases} K_0^+ : \text{tot. real} \\ \text{Leopoldt's conj} \\ \text{CFT} \end{cases} \rightsquigarrow (\sigma-1) \nmid \text{char}(X_K^+)$
(no trivial zero!)

III. Analytic class number formula.

Then, enough to show.

Thm' $\psi : \text{even} \in \hat{\Delta} \neq 1$

$$p^a (\sigma-1)^b \text{char}_{\Lambda_\psi}((E_\infty^+/C_\infty^+)_\psi)$$

$$\subseteq \text{char}_{\Lambda_\psi}(X_K^+)_\psi$$

└

For some $a, b \in \mathbb{Z}_{\geq 0}$