大下丘　2012年4月4日

Second talk

Proof of IMC for abelian fields by using Euler system method II

Today

§2. Euler systems of circular units

§3. Proof of main theorem $\begin{cases} \text{preliminaries} \\ \text{main step} \end{cases}$

§2. observation

odd

$n \in \mathbb{Z}_{>1}$, ℓ : prime number

$$N_{\mathbb{Q}(\mu_{n\ell})/\mathbb{Q}(\mu_n)}(1-\zeta_{n\ell}) = \begin{cases} 1-\zeta_n & \ell \mid n \\ \dfrac{1-\zeta_n}{1-\zeta_n^{1/\ell}} = (1-\zeta_n)^{1-\mathrm{Fr}_\ell^{-1}} & \end{cases}$$

arith Frob in $\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$

For each prime number ℓ, we put

$\mathbb{Q}(\ell)$: the p-ray class field of $\mathbb{Q}$ mod ℓ

For $n = \ell_1 \times \cdots \times \ell_r$ (square-free)

$\mathbb{Q}(n) = \mathbb{Q}(\ell_1) \cdots \mathbb{Q}(\ell_r)$

$H_n := \mathrm{Gal}(\mathbb{Q}(n)/\mathbb{Q}) \cong H_{\ell_1} \times \cdots \times H_{\ell_r}$

We put $K_m^+(n) := K_m^+ \mathbb{Q}(n)$

$\left(\begin{array}{l} \text{Assume } (n, \text{cond } K_0^+/\mathbb{Q}) = 1 \\ \Rightarrow H_n \simeq \mathrm{Gal}(K_m^+(n)/K_m^+) \end{array} \right)$

In this talk,

we call a family $\{ c_m(n) \in \mathcal{O}_{K_m^+(n)}[1/p]^\times \}$

$(m \geq 0, (n, f) = 1)$

with the following condition, the Euler system

(E1) $N_{K_m^+(n\ell)/K_m^+(n)}(C_m(n\ell)) = C_m(n)^{1-Fr_\ell^{-1}}$

$(Fr_\ell \in Gal(K_m^+(n)/\mathbb{Q})$ arith Frobenius $)$

(E2) $N_{K_{m+1}^+(n)/K_m(n)}(C_{m+1}(n)) = C_m(n)$

((E3) Let ℓ : prime number s.t $\ell \nmid fn$
Then, for $\forall \lambda$: prime of $K_m^+(n\ell)$ above ℓ,
$C_m(n\ell) \equiv C_m(n)^{Fr_\ell^{-1}} \mod \lambda$)

If $p \nmid \# \Delta$, the condition (E3) follows from (E1) & (E2)

By "observation" & definition of circular units,
for $\forall C(1) = (C_m)_{m \geq 0} \in C_\infty^+$
$\exists C = \{C_m(n) \in C(K_m^+(n))\}$ Euler system.
s.t $C_m(1) = C_m$ for $\forall m \geq 0$.

$m \in \mathbb{Z}_{>0}$, M : p-power > 1.

$$\mathcal{S}_{m,M} := \left\{ n \in \mathbb{Z}_{>0} \,\middle|\, \begin{array}{l} ① \text{ sq. free} \\ ② \ \forall \ell \mid n \text{ prime, } \ell \equiv 1 \mod M \\ ③ \ \ell \text{ splits completely in } K_m^+ \end{array} \right\} \cup \{1\}$$

For each $\ell \in \mathcal{S}_{m,M}$: prime,
we fix $\sigma_\ell \in H_\ell$: generator
Put $M' := \# H_\ell$
$Hom(I_{\mathbb{Q}_\ell}, \mu_{M'}) \xrightarrow{\sim} (\mathbb{Q}_\ell^{ur})^\times/M \xrightarrow{\sim} \mathbb{Z}/M'$

Then, we have $H_\ell \xrightarrow{\sim} \mu_M,\ \sigma_\ell \mapsto \zeta_M$

For $n = l_1 \times \cdots \times l_r \in \mathcal{S}_{m,M}$

$$D_{l_i} := \sum_{1 \le k \le \# H_{l_i} - 1} k\sigma_{l_i}^{k} \in \mathbb{Z}[H_{l_i}]$$

$$\cap\!\!\mid$$

$$D_n := \prod_{i=1}^{r} D_{l_i} \in \mathbb{Z}[H_n]$$

Let $C = \{ C_m(n) \in K_m^+(n)^\times \}$ Euler system

Prop.-Def (Kolyvagin derivative class)

$$\begin{array}{ccc} (K_m^+)^\times/M & \xrightarrow{\sim} & \left(K_m^+(n)^\times/M\right)^{H_n} \\ & & \cup \\ \exists!\ K_{m,M}(n,C) & \dashrightarrow & C_m(n)^{D_r} \\ \| & & \\ (K_n) & & \end{array}$$

Rem Δ is isom. since

$$H^1\left(K_m^+(n)/K_m^+,\ \underbrace{H^0(K_m^+(n), \mu_n)}_{=0}\right) = 0$$

real

Rem2 We can define $K_{m,M}(n,C)$ even when $p=2$

Localization & finite-singular maps

Let $\ell \in \mathcal{S}_{m,M}$ prime number

Put $\mathcal{I}_m(\ell) := \mathrm{Div}(\mathcal{O}_{K_m^+} \otimes \mathbb{Z}_\ell) \simeq \bigoplus_{\lambda | \ell} \mathbb{Z}$

$\hookrightarrow$ a free $\underbrace{\mathbb{Z}[\mathrm{Gal}(K_m^+/\mathbb{Q})]}_{R_m^0}$-module

ℓ splits in $K_m^+/\mathbb{Q}$ of rank one.

$R_m := R_m^0 \otimes_{\mathbb{Z}} \mathbb{Z}_p$

We will define two R_n-homs $(K_m^+)^\times/M \to l_n(\ell)/M$ ④

① We write $\mathrm{div}_\ell : (K_m^+)^\times \longrightarrow l_m(\ell)$

then, we define

$$[\ ,\ ]^\ell_{m,n} : (K_m^+)^\times/M \longrightarrow l_m(\ell)/M$$

(the map induced by div_ℓ)

② Let $\mathrm{rec}^\ell : \mathcal{O}_\ell^\times \longrightarrow G_{\mathcal{O}_\ell}^{ab} \twoheadrightarrow H_\ell$

$$\ell \longmapsto \mathrm{Fr}_\ell$$

loc. rec. map in LCFT

We define,

$$(K_m^+)^\times/M \xrightarrow{\mathrm{diag}} \bigoplus_{\lambda \mid \ell} \mathcal{O}_\ell^\times/M \longrightarrow \bigoplus_{\lambda \mid \ell} H_\ell/M$$

$K_\lambda^\times$ = $\mathcal{O}_\ell^\times$

$\lambda \mid \ell$ places of K_m^+

$(0, \cdots, \sigma_\ell, \cdots, 0)$

$\mathrel{\rotatebox[origin=c]{90}{$\mapsto$}}$

$\xrightarrow{\sim} l_m(\ell)/M \qquad \lambda \bmod M$

$\phi^\ell_{m,M}$ "finite sing map"

<u>Prop 1</u> Let $n\ell \in \mathcal{S}_{m,M}$, ℓ : prime number

$\kappa_{m,M}(n, c)$: Kolyvagin derivative class

κ_n = $\kappa_{m,M}(n, c)$

<u>Then</u>,

① $[\kappa_{m,M}(n)]^\ell_{m,M} = 0$

② $[\kappa_{m,M}(n\ell)]^\ell_{m,M} = \phi^\ell_{m,M}(\kappa_{m,M}(n))$

§3. Proof of Thm'

I. Preliminary lemmas from control thms.

For $m \in \mathbb{Z}_{\geq 0}$ M : p-power > 1 (fix)

$$\left.\begin{aligned} R_m &:= \mathbb{Z}_p[\mathcal{G}_m^+] \\ R_{m,M} &:= R_m / M \end{aligned}\right) \longleftarrow \Lambda\text{-alg}$$

Let $\psi \in \hat{\Delta}$: even $\neq \mathbb{1}$

Put $\cdot \operatorname{char}_{\Lambda_\psi}((E_\infty^+/C_\infty^+)_\psi) = (H_\psi)$

$\cdot$ $\tau : X_\psi \xrightarrow[\text{pseudo-isom.}]{} \bigoplus_{i=1}^{r} \Lambda_\psi / (g_{i,\psi})$

$$\prod_{i=1}^{r} g_{i,\psi} = \operatorname{char}_{\Lambda_\psi}(X_{K,\psi})$$

Then we have the following lemma.

<u>Lemma 2</u>

$\exists\, C_1 \in \mathbb{Z}_{>0}$ constant (indep. of n)

$\exists\, \vartheta_m : E_{m,\psi}^+ \longrightarrow R_{m,\psi}$: $R_{m,\psi}$-module hom.

s.t $\begin{cases} (\gamma - 1)^2 p^{C_1} \boxed{H_\psi R_{m,\psi}} \subseteq \vartheta_m(I_m C_{\infty,\psi}) \\ p^{C_1} \vartheta_m(I_m C_{\infty,\psi}^+) \subseteq H_\psi \cdot R_{m,\psi} \end{cases}$

<u>Lemma 3</u> $R_{m,\psi} = \Lambda_\psi / (\gamma^{p^m} - 1)$

$\exists\, C_2 \in \mathbb{Z}_{\geq 0}$ constant

$\exists\, \tau_m : A_{K_m^+,\psi} \longrightarrow \bigoplus_{i=1}^{r} R_{m,\psi} / (g_{i,\psi})$

s.t $p^{C_2} \operatorname{coker}(\tau_m) = 0$

Choose & fix $c_i \in A_{K_m^+, \psi}$

s.t $\tau_m(c_i) = (0, \cdots, p^{c_2}, \cdots, 0)$

II. <u>main step</u>

Here, we fix m & M $(M \gg 0)$

By Lemma 2, $\exists\, \xi = (\xi_{m'}) \in C_\infty^+$

s.t $\partial_m(\xi_{m,\psi}) = (\sigma-1)^2 p^{c_1} H_\psi$ in $R_{m,\psi}$

Then, we take an E.S.

$C = \{ C_{m'}(n) \in C(K_m^+(n)) \}$ s.t $\boxed{\{C_{m'}(1)\}_{m'} = \xi}$

In the next talk,

we will construct, inductively, prime ideals,

λ_i of K_m^+ satisfying the following (a)-(c)

(a) $[\lambda_i] = p^2 c_i$

(b) For l_i prime number below λ_i

$l_i \in \mathcal{S}_{m,M}$

and $l_i \neq l_j$ $(i \neq j)$

(c) In $R_{m,M,\psi}$, we have

$\cdot\ V_{\lambda_1}(K_{l_1})_\psi = (\text{unit}) \cdot \#\Delta \cdot p^{2+c_1} \times H_\psi$

$\cap (\mathbb{Z}/M)^\times$

$\cdot\ \partial_{i-1,\psi} \boxed{V_{\lambda_i}(K_{l_1 \times \cdots \times l_i})_\psi}$

$= (\text{unit}) \times \#\Delta \times p^{2+c_2} \cdot (\sigma-1)^{2^{i-1}} \times \boxed{V_{\lambda_{i-1}}(K_{l_1 \times \cdots \times l_{i-1}})_\psi}$

$n_i = l_1 \times \cdots \times l_i$

$$V_{\lambda_i} : (K_m^+)^\times/M \xrightarrow{[\ ]_{m,M}^{l}} \mathcal{U}_m(l)/M \simeq R_{m,M}$$

$$\lambda_i \longmapsto 1$$

Existence of $\{\lambda_i\}$ with (a) ~ (c)

$\Rightarrow$ <u>Thm′</u>

In $R_{m,M,\psi}$

$$\mathfrak{d}_{1,\psi} \times \cdots \times \boxed{\mathfrak{d}_{r,\psi} \times V_{\lambda_{r+1}}(K_{n_{r+1}})_\psi} \quad (c)$$

$$= \mathfrak{d}_{1,\psi} \times \cdots \times \mathfrak{d}_{r,\psi} \times \#\Delta^+ \times p^{2+c_2}(r-1)^{2^r} \boxed{V_{\lambda_r}(K_{n_r})_\psi}$$

$$= \cdots = (\#\Delta)^r p^{r(2+c_2)} (\gamma-1)^{2^{r+1}-1} \times V_{\lambda_1}(K_{l_1})_\psi$$

$$= p^a \times (\gamma-1)^b \times H_\psi \quad \text{(up to unit)}$$

Note r, c_1 & c_2 are indep. of m & M.

$$\left(\begin{array}{c} \Rightarrow \text{RHS is "coherent"} \\ \text{by } R_{m',M'} \longrightarrow R_{m,M} \end{array} \right)$$

Taking limit $\varprojlim_{m,M} R_{m,M\psi} = \Lambda_\psi$

$$\prod_{i=1}^{r} \mathfrak{d}_{i,\psi} \text{ divides } p^a(\gamma-1)^b H_\psi$$