大下 ? 2012年4月5日

Third talk

Proof of IMC for abelian fields via Euler systems of circular units

In this talk, we will construct a sequence of prime ideals $\{\lambda_i\}_{i=1}^{r+1}$ of K_m with conditions

(a) $[\lambda_i]_\psi = C_i$

(b) For l_i prime number below λ_i, $l_i \in \mathcal{L}_{min}$ & $l_i \neq l_j$ $(i \neq j)$

(c) In $R_{m,M,\psi}$

$$\nu_{\lambda_1}(\kappa_{l_1})_\psi = (\text{unit}) \cdot \# \Delta \cdot p^{c_1} \cdot H_\psi$$

$$g_{i-1} \nu_{\lambda_i}(\kappa_{\underbrace{l_1 \times \cdots \times l_i}_{n_i}})_\psi$$

$$= (\text{unit}) \times \# \Delta \cdot p^{c_2} (\gamma - 1)^{2^{i-1}} \times \nu_{\lambda_{i-1}}(\kappa_{n_{i-1}})$$

This argument is divided into 3-parts

I. Čebotarev's density thm.
(hom $\leadsto$ prime)

II "Global duality"
(prime $\rightarrow$ hom)

III Induction arguments via Euler system.

I. Lemma from Čebotarev's density thm.

<u>Lemma 4</u>

$m \in \mathbb{Z}_{\geq 0}$, M: p-power order

Suppose the following data are given

- $c \in A^+_{K_m}$
- $W \subseteq (K_m^+)^\times/M$ a finite $R_{m,M}$-submodule
- $\Phi : W \longrightarrow R_{m,M}$: an $R_{m,M}$-hom

Then $\exists$ infinitely many λ primes of K_m^+ satisfying the following 4 conditions;

(1) $[\lambda] = c$ in $A^+_{K_m}$

(2) For l : prime number below λ, $l \in \mathcal{S}_{m,M}$

(3) $W \subseteq \operatorname{Ker}[\cdot]_{m,M}$

(4) $\exists\, u \in (\mathbb{Z}/M)^\times$ such that for $\forall w \in W$

$$\phi^l_{m,M}(w) = u \cdot \Phi(w) \cdot \lambda$$

p-Hilbert class field

$$\begin{array}{ccc} & & K'' := K'(W^{1/M}) \\ & & | \\ H & & K' := K_m^+(\mu_m) \\ & \diagdown \quad \diagup & \\ & K_m^+ & \end{array}$$

Note K'' and H are lin. disjoint over K_m^+.

We will take a "good" element of $\mathrm{Gal}(HK''/K_m^+)$.

We have an isom. (as groups)

$$\mathrm{Hom}_{R_{m,M}}(W, R_{m,M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}/M}(W, \mu_M)$$
$$f \longmapsto p \circ f$$

where $p : R_{m,M} \longrightarrow \mu_M$

$$\sum_\sigma a_\sigma \sigma \longmapsto \zeta_M^{a_1}$$

Note

complex conj acts by -1. complex conj acts by $+1$

$$0 \longrightarrow H^1(K'/K_m^+, \mu_M) \longrightarrow H^1(K_m^+, \mu_M) \longrightarrow H^1(K', \mu_M)$$

$$H^1(K'/K_m^+, \mu_M) = 0$$

So we regard $W \subseteq K'^\times/M$

By Kummer theory, $\mathrm{Hom}(W, \mu_M) \xrightarrow{\sim} \mathrm{Gal}(K''/K')$

$$\begin{array}{ccc} \Psi & & \Psi \\ p \circ \Phi & \longmapsto & \tau \end{array}$$

Let $\sigma \in \mathrm{Gal}(K''/K_m^+)$ be $\begin{cases} \sigma|_H = c & \text{in } \mathrm{Gal}(H/K_m^+) \overset{\shortparallel}{A}_{K_m}^+ \\ \sigma|_{K''} = \tau \end{cases}$

By Čebotarev density thm,

$\exists$ infinitely many primes λ of K_m^+

such that

$\begin{cases} \lambda \text{ is unr. in } K''/K_m^+ \\ (\lambda, K''/K_m^+) = \text{conjgacy class of } \sigma \end{cases}$

one can check that such λ satisfies (1)–(4)

II Lemma from "Global duality"

Essence

"$Cl_{K_m^+}$ = "ideal group"/"principal"

"a kind of global duality"

<u>Lem</u> $\psi \in \hat{\Delta}$: even char. M : p-power $n = l_1 \times \cdots \times l_i \in \mathcal{S}_{m,M}$

<u>Set</u> ◎ $l := l_i$, $\lambda | l$: a prime of K_m^+

$c := [\lambda] \in A_{K_m^+, \psi} =: A_\psi$

◎ $B_\psi \subseteq A_\psi$: the $R_{m,\psi}$-submodule generated by the images of primes above $l_1, \cdots, l_{i-1}$.

◦ Let $x \in (K_m^+)^\times/M$ s.t. $[x]_{m,M}^q = 0$ for $\forall q \nmid n$.

and put $W := \langle x \rangle_{R_m} \subseteq K_m^{+\times}/M$

Assume

$\exists\, E,\ \underline{\underline{g}},\ \eta \in R_m$ satisfying

p-power $\quad \overset{\psi}{g_{i,\psi}} \quad$ r-1-power

(i) $E \cdot \mathrm{Ann}_{R_{m,\psi}}\left(C \text{ in } (A/B)_\psi\right) \subseteq g_\psi R_{m,\psi}$

(ii) $\#\left(R_{m,\psi}/(g_\psi)\right) < \infty$

(iii) $M \geq \# A_\psi \times \#\left(\eta \cdot \left(\frac{\ell_m(\ell)/M}{W}\right)_\psi\right)$

Then $\exists\, \Phi : \overbrace{W_\psi \longrightarrow R_{m,M,\psi}}^{R_m\text{-hom.}}$ such that

$$\left(g \cdot \Phi(x) \cdot \lambda\right)_\psi = E \cdot \eta \cdot [x]^{\ell}_{m,M}$$

pf $\quad$ Put $\beta \in (K_m^+)^\times$ a lift of $x \in W$

By $\mathrm{div}_\ell(X) = V_\lambda(\beta) \cdot \lambda$ and the definition of X,

$V_\lambda(\beta) \cdot C = 0$ in $(A/B)_\psi$

$\overset{(i)}{\rightsquigarrow} E \cdot V_\lambda(\beta)_\psi \in g_\psi R_{m,\psi}$

By (ii) $\quad R_{m,\psi} \xrightarrow{\times g_\psi} R_{m,\psi}$: injective

so $\exists!\, \underline{\underline{\delta}} \in R_{m,\psi}$ s.t. $E \cdot V_\lambda(\beta)_\psi = g_\psi \boxed{\delta}$

want to define $\Phi : W_\psi \longrightarrow R_{m,M,\psi}$

$$\rho \cdot x \longmapsto \eta \cdot \rho \cdot \delta \quad (\rho \in R_{m,\psi})$$

We have to show "well-definedness".

Assume $\rho X = 0$

i.e. $\beta^\rho = y^M \in (K^\times/M)_\psi$

i.e. $\beta^{\sigma} = y^{M} \in (K_m^{+\times} \otimes \mathbb{Z}_p)_\psi$

So $\rho[x]^{\ell}_{m,M} = 0$ in $\mathcal{I}_m(\ell)/M$

By (iii) $\underbrace{M\cdot(\#A_\psi)^{-1}}_{\substack{\cap \\ \mathbb{Z}}} \cdot \eta \cdot \left(\mathcal{I}_m(\ell)/M\right) \subseteq R_{m,M,\psi}\cdot [x]^{\ell}_{m,M,\psi}$

Then, $\rho\eta$ annihilates $M\cdot(\#A)^{-1}\boxed{(\mathcal{I}_m(\ell)/M)_\psi}$

$\therefore\ \rho\eta \in (\#A)\cdot R_{m,M,\psi}$

In $(A/B)_\psi$ $\quad \eta\, div(y) := \eta\cdot \sum_{q:\text{prime number}} div^{q}(y)$

$$= \eta\cdot[y]^{\ell}_{m,M,\psi} + \eta \underbrace{\sum_{j=1}^{i-1}[y]^{\ell_j}_{m,M,\psi}}_{\substack{= \\ 0}\ \in B}$$

divided by $\#A_\psi$

$$+ \sum_{q\nmid n} \frac{\rho\eta}{M}[x]^{q}_{m,M,\psi} \in \mathcal{I}_m(q)$$

$\boxed{\eta\rho\delta = 0}$

$\therefore\ \eta[y]^{\ell}_{m,M,\psi} = 0 \overset{(i)}{\rightsquigarrow} E\eta\cdot\kappa_\lambda(y) \in \partial R_{m,\psi}$.

III. Induction argument

Here we construct $\{\lambda_i\}$ with (a) – (c)

<u>Rem</u> (elementary)

N: Λ-module, $\psi \in \hat{\Delta}$ character.

$\pi: N \longrightarrow N_\psi$ a natural projection.

$\Rightarrow\ \exists\ \varepsilon_\psi : N_\psi \longrightarrow N$ s.t $\pi\varepsilon_\psi = |\Delta|\cdot id$.

<u>Construction</u>

$i=1$ Apply Lemma 4 to

$c =$ (pre-image of) $c_1 \in A_\psi$

$W := \mathcal{O}_{K_m^+}^\times / M$

$\Phi : W \longrightarrow E_{m,\psi}/M \xrightarrow{\vartheta_{m,\psi}} R_{m,M,\psi} \xrightarrow{\varepsilon_\psi} R_{m,M}$

$\leadsto {}^\exists \lambda_1$: prime of K_m^+ $(\lambda_1 | \ell_1)$

$$\begin{cases} [\lambda_1]_{\psi_1} = c_1 \\ \ell_1 \in \mathcal{L}_{m,M} \\ \phi^{\ell_1}|_W = (\text{unit})\,\Phi \end{cases}$$

By Prop. 1

$$\nu_{\lambda_1}(\kappa_{\ell_1})\lambda_{1,\psi} = \phi_{m,M}^{\ell_1}(C_m(1))_\psi$$

choice of E.S $\;= (\text{unit}) \times \#\Delta \cdot (\gamma-1)^2 \cdot p^{c_1} \times H_\psi \cdot \lambda_{1,\psi}$

$\underline{i-1 \leadsto i}$

Assume we have $\lambda_1, \cdots, \lambda_{i-1}$,

and now we construct λ_i

choose $h \in \mathbb{Z}_{\geq 1}$ s.t.

$$h \geq \begin{cases} \#R_{m,\psi}/(H_\psi) \\ \#R_m/\#\Delta \\ \#R_{m,\psi}/(p) \end{cases}, \quad \text{and put } M := \#A \times h$$

By (C),

$$\underline{\nu_{\lambda_{i-1}}(\kappa_{n_{i-1}})_\psi} \;\Big|\; p^{c_2}\#\Delta(\gamma-1)^{2^{i-1}}\, \nu_{\lambda_{i-2}}(\kappa_{n_{i-2}}) \Big| \cdots \Big| \underbrace{p^{c_2(i-2)}\, p^{c_1} \times (\#\Delta)^{i-1}(\gamma-1)^{2^{i-1}} H_\psi}_{=d}$$

? ?

Put $N := (\sigma-1)^{2^{i-1}} \left(I_m(l_i)/M \right) \Big/ \underbrace{\langle \overset{x}{[K_{n_i}]^{l_i}_{m,M,\psi}} \rangle}_{M}$

N annihilated by $\mathcal{H}_\psi$

Choice of M $\#N \le p^{i_m} M \cdot C \, \#A_\psi$

We set

$$\begin{cases} n = n_{i-1}, & l = l_{i-1}, \quad \lambda = \lambda_{i-1} \\ g = g_{i-1,\psi}, & x = K_{n_{i-1}}, \quad E = p^{i_2} \\ \eta = (\sigma-1)^{2^{i-1}} \end{cases}$$

Apply Lemma 5:

$\exists\, \Phi_{i,\psi} : W_{i,\psi} \longrightarrow R_{m,M}$

s.t. $g_{i-1,\psi}\, \Phi_i (K_{n_{i-1}})_\psi = p^{C_2} (\sigma-1)^{2^{i-1}} \cdot \kappa_{\lambda_{i-1}} (K_{n_{i-1}})$

Apply $\boxed{\text{Lemma 4}}$ for $\begin{cases} C = C_i \\ W = W_i \\ \Phi = \varepsilon_\psi \cdot \Phi_{i,\psi} \end{cases}$

Then $\exists\, \lambda_i$: prime of K_m^+

s.t. $\begin{cases} [\lambda_i] = C_i \\ \lambda \nmid n_{i-1}, \quad \lambda \in \mathcal{S}_{m,M} \\ \phi_{m,M}(w) = (\text{unit}) \times \Phi_i(w)_\psi \quad \text{for } w \in W. \end{cases}$

Then we obtain $\phi^{l_i} = \Phi_i$

$$g_{i-1} [K_{n_{i-1}}]^{l_i}_{m,M,\psi} \overset{g_{i-1}\Phi_i = [\ ]^{l_i} g_{i-1}}{=} g_{i-1}\, \phi^{l_i} (K_{n_{i-1}})_\psi \quad \text{with} \quad g_{i-1}\,\phi^{l_i}(K_{n_{i-1}}) = p^{\circ}(\sigma-1)^{\circ} \kappa_{\lambda_{i-1}}(K_{n_{i-1}})$$

$\lambda \lrcorner$

Hence we obtain λ_i.

We complete the proof //