保型L関数の非消滅性の現状

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1 The Standard L-functions of $\mathrm{GL}(n)$

- F : an algebraic number field.
- $\Sigma_F = \Sigma_{\text{fin}} \cup \Sigma_\infty$: the set of places of F.
- $\mathfrak{O}_F$: the ring of integers of F.
- $\mathbb{A}_F$: the ring of adeles of F.
- $\mathcal{A}_0(G_n)$: the set of irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. ($G_n := \mathrm{GL}(n)$)
- $\mathcal{A}_0^u(G_n)$: the set of $\pi \in \mathcal{A}_0(G_n)$ whose central character ω_π is unitary.

1.1 The Vogan-Tadic classification

([33], [34], [37], a convenient reference is [14])

Fix $v \in \Sigma_F$**. Set**

$$\Pi^u(G_n(F_v)) := \{\textbf{irreducible smooth unitary rep's of } G_n(F_v)\}/\sim,$$

$$\Pi^u_{\mathrm{d}}(G_n(F_v)) := \{\delta \in \Pi^u(G_n(F_v)) | \, \delta \textbf{ is square integrable}\},$$

$$\Pi^u_{\mathrm{sc}}(G_n(F_v)) := \{\sigma \in \Pi^u(G_n(F_v)) | \, \sigma \textbf{ is supercuspidal}\}(\textbf{when } v < \infty)$$

- **The unramified twist of** $\pi \in \Pi(G_n(F_v))$ **by** $s \in \mathbb{C}$ **is defined as**

$$\pi(s) \, : \, g \mapsto |\det g|^s \pi(g).$$

• **(Discrete series)** $\Pi^u_{\mathrm{d}}(G_n(F_v)) = \{Q(\sigma,d) \,|\, \sigma \in \Pi^u_{\mathrm{sc}}(G_{n/d}(F_v)),\ d|n\}$**,**

where $m = n/d$**,**

$$I_{m,\dots,m}\left(\sigma\left(\tfrac{1-d}{2}\right), \sigma\left(\tfrac{3-d}{2}\right), \dots, \sigma\left(\tfrac{d-1}{2}\right)\right) \underset{\textbf{unique quotient}}{\longrightarrow} Q(\sigma,d).$$

• **(Speh module) For** $\delta \in \Pi^u_{\mathrm{d}}(G_m(F_v))$ **and** $d \geqslant 2$**, the module** $\mathrm{sp}(\delta,d)$ **is an element of** $\Pi^u(G_{md}(F_v))$ **defined as**

$$I_{m,\dots,m}\left(\delta(\tfrac{d-1}{2}), \delta(\tfrac{d-3}{2}), \dots, \delta(\tfrac{1-d}{2})\right) \underset{\textbf{the Langlands quotient}}{\longrightarrow} \mathrm{sp}(\delta,d).$$

• **(Complementary series) For** $\delta \in \Pi^u_{\mathrm{d}}(G_m(F_v))$**,** $d \in \mathbb{N}$ **and** $\alpha \in (0,1/2)$**, set**

$$u(\delta,d,\alpha) := I_{md,md}\Big(\mathrm{sp}(\delta,d)(\alpha), \mathrm{sp}(\delta,d)(-\alpha)\Big),$$

where we set $\mathrm{sp}(\delta,1) = \delta$**.**

• $\mathcal{B}(n)$ **: all possible rep's of** $G_n(F_v)$ **among all those** δ, $\mathrm{sp}(\delta, d)$ **and** $u(\delta, d, \alpha)$ **constructed above.**

THEOREM (Vogan-Tadič)

The full unitary dual of $G_n(F_v)$ **is given as**

$$\Pi^u(G_n(F_v)) = \{ I_{n_1,\ldots,n_r}(\sigma_1, \ldots, \sigma_r) |\, \sigma_j \in \mathcal{B}(n_j)\, (1 \leqslant j \leqslant r) \}/ \sim .$$

The multiset $\{\sigma_1, \ldots, \sigma_r\}$ **determines the class of** π **uniquely.**

• **generic** $\Leftrightarrow$ **Blue blocks** $\notin \{\sigma_j\}$.

• **tempered** $\Leftrightarrow$ **Blue blocks & Green blocks** $\notin \{\sigma_j\}$.

1.2 The standard L-functions

- $\pi \cong \otimes_{v\in\Sigma_F}\pi_v \in \mathcal{A}_0(\mathrm{GL}_n)$, **the standard L-function is defined as**

$$L(s,\pi) := \prod_{v\in\Sigma_F} L(s,\pi_v) \quad (\textbf{abs. conv. on } \operatorname{Re} s \gg 0).$$

The local factor $L(s,\pi_v)$ is described as follows:

(i) If $I(\delta_1,\dots,\delta_r) \to \pi_v$ **(Langlands quoteint) with** $\delta_j \in \Pi^u_{\mathrm{d}}(G_{n_j}(F_v))$,

$$L(s,\pi_v) = \prod_{j=1}^{r} L(s,\delta_j).$$

(ii) If $v \in \Sigma_{\mathrm{fin}}$ **and** $\delta \cong Q(\sigma, d)$ **with** $\sigma \in \Pi_{\mathrm{sc}}(G_m(F_v))$ $(n = dm)$**, then**

$$L(s, \delta) = L\left(s + \tfrac{d-1}{2}, \sigma\right).$$

If $v \in \Sigma_\infty$ **and** $\delta \in \Pi^u_{\mathrm{d}}(G_m(F_v))$ **(**$m = 1, 2$**),**

$$L(s, \delta) = \begin{cases} \Gamma_{\mathbb{R}}\left(s + \nu + \epsilon\right), & (\delta \cong \chi_{\nu,\epsilon}\,(\nu \in \mathbb{C},\, \epsilon \in \{0, 1\}), \\ \Gamma_{\mathbb{C}}\left(s + \frac{l}{2} + \nu\right), & (\delta \cong D_{\nu,l}\,(\nu \in \mathbb{C},\, l \in \mathbb{N}). \end{cases}$$

(iii) Let $v \in \Sigma_{\mathrm{fin}}$**. For** $\sigma \in \Pi_{\mathrm{sc}}(G_n(F_v))$**,**

$$L(s, \sigma) = \begin{cases} (1 - q_v^{-s})^{-1} & (n = 1,\, \sigma = \mathbf{1}), \\ 1 & (\textbf{otherwise}). \end{cases}$$

Basic properties of the standard L

Let $\pi \in \mathcal{A}_0^u(G_n)$ **with** $n > 1$**. Then**

(1) $L(s,\pi) = \prod_v L(s,\pi_v)$ **conv. abs. on** $\mathrm{Re}\, s > 1$**.**

(2) $L(s,\pi)\,(\mathrm{Re}\, s > 1)$ **has a holomorphic continuation to** $\mathbb{C}$**.**

(3) $L(s,\pi)$ **and** $L(1-s,\pi^\vee)$ **satisfies**

$$L(s,\pi) = W(\pi)(D_{F/\mathbb{Q}}^n \mathrm{N}(\mathfrak{q}_\pi))^{1/2-s} L(1-s,\pi^\vee), \tag{1}$$

where $\mathfrak{q}_\pi$ **is the conductor of** π **and** $W(\pi) = \epsilon(\frac{1}{2},\pi) \in \mathbb{C}^1$ **is the root number.**

(3) The holomorphic function $L(s,\pi)$ **is bounded on any vertical strip** $\{\sigma \in \mathbb{C} |\, a < \mathrm{Re}\, s < b\}$**.**

THEOREM (Jacquet-Shalika [9])

Let $\pi \in \mathcal{A}_0^u(G_n)$. Then $L(s,\pi) \neq 0$ on the edge of the critical strip $\mathrm{Re}\, s = 0, 1$.

THEOREM (Shalika)

Any $\pi \in \mathcal{A}_0(G_n)$ is globally generic.

If $\pi \cong \otimes_v \pi_v$, then, for each $v \in \Sigma_F$, π_v is a generic representation of $G_n(F_v)$, i.e.,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_v(F_v)}(\pi_v, C^\infty \mathrm{Ind}_{U_n(F_v)}^{G_n(F_v)}(\theta)) = 1,$$

where $U_n(F_v)$ is the upper-triangular maximal unipotent subgroup and θ is the character $\psi_v(\sum_{j=1}^{n-1} u_{jj+1})$ of $U_v(F_v)$.

A consequence of genericity and the Vogan-Tadic classification

Suppose $\pi \in \mathcal{A}_0^u(G_n)$**. Then** $\exists S \subset \Sigma_F$ **(finite set) s.t.**

$$\pi_v^{\mathrm{GL}_n(\mathfrak{O}_v)} \neq \{0\} \qquad (\forall v \in \Sigma_{\mathrm{fin}} - S).$$

- **For** $v \in \Sigma_{\mathrm{fin}} - S$**,**

$$\pi_v \cong I_{1,\ldots,1}\left(\{\chi_j\}_{j=1}^a, \{\chi_j(\nu_i), \chi_j(-\nu_i)\}_{i=a+1}^n)\right)$$

with $\chi_j \in \widehat{F_v^\times}$ **(unramified), and** $\nu_j \in (0, 1/2)$**.**

- **For** $v \in S$**,**

$$\pi_v \cong I_{n_1,\ldots,n_r}\left(\{\delta_j\}_{j=1}^a, \{\delta_j(\nu_j), \delta_j(-\nu_j)\}_{i=a+1}^r\right)$$

with $\delta_j \in \Pi_{\mathrm{d}}^u(G_{n_j}(F_v))$ **and** $\nu_j \in (0, \frac{1}{2})$**.**

Remark : The generalized Ramanujan conj. for $\pi \Leftrightarrow \{\nu_j\}$ **is empty (**$\forall v \in \Sigma_F$)**.**

Holomorphy of local L-factors

Let $\pi \cong \otimes_v \pi \in \mathcal{A}_0^u(G_n)$**.**

(i) For $v \in \Sigma_F$**, there exists** $\theta_v > 0$ **such that** $L(s, \pi_v)$ **has neither zeros nor poles on** $\mathrm{Re} > 1/2 - \theta_v$**.**

(ii) If π_v **is tempered, then** $L(s, \pi_v)$ **has neither zeros nor poles on** $\mathrm{Re}\, s > 0$**.**

A remark on trivial zeros

For a finite set $S \subset \Sigma_F$,

$$L^S(s,\pi) := \prod_{v \notin S} L(s,\pi_v) \quad (\mathrm{Re}\, s > 1).$$

Let $s_0 \in \mathbb{C}$ with $\frac{1}{2} \leqslant \mathrm{Re}\, s_0 < 1$. Then

$$L(s_0,\pi) \neq 0 \quad \Longleftrightarrow \quad L^S(s_0,\pi) \neq 0$$

If $\pi_S = \otimes_{v\in S}\pi_v$ is tempered, the same is true for $0 < \mathrm{Re}\, s_0 < 1$.

This follows from the local holomorphy of L-factors. Due to the lack of the gamma factors, $L^\infty(s,\pi)$ has more zeros than $L(s,\pi)$; the extra zeros are called the trivial zeros of $L^\infty(s,\pi)$ (*cf.* $s = -2, -4, \ldots$ for $\zeta(s)$.)

2 The problem

- $\mathbb{X}$: the set of continuous characters $\chi : \mathbb{A}_F^\times / F^\times F_\infty^\times \to \mathbb{C}^1$.
- $\mathfrak{q}_\chi$: the conductor of $\chi \in \mathbb{X}$.
- For $\mathfrak{q} \subset \mathfrak{O}_F$, $\mathbb{X}(\mathfrak{q}) := \{\chi \in \mathbb{X} |\, \mathfrak{q}_\chi = \mathfrak{q}\,\}$.

Example : When $F = \mathbb{Q}$ and $q \in \mathbb{N}$, $\mathbb{X}(q)$ is the primitive even Dirichelet characters modulo q.

PROBLEM (nonvanishing of twisted L-values)

- $\mathcal{A} \subset \mathcal{A}_0^u(G_n)$: a class of cuspidal representations
- $X \subset \mathbb{X}$: a (large) subset.

Given a $\pi \in \mathcal{A}$ and a point $s_0 \in \mathbb{C}$,

(1) does there exist $\chi \in X$ such that $L(s_0, \pi \otimes \chi) \neq 0$?

(non-vansihing)

(2) For $\pi \in \mathcal{A}$, how the cardinarity

$$\#\{\chi \in X \cap \mathbb{X}(\mathfrak{q}) | L(s_0, \pi \otimes \chi) \neq 0\}$$

grows as $\mathrm{N}(\mathfrak{q}) \to \infty$? (quantitative non-vanishing)

$\boxed{\text{Remarks}}$:

- This is non-trivial only when $0 < \operatorname{Re} s_0 < 1$.
- This is a problem on the classes $\mathcal{A}$ and X (not on individual χ). (The nature of $\mathcal{A}$ and X is of great relevance.)
- GRH predicts $L(s_0, \pi \otimes \chi) \neq 0$ unless $\operatorname{Re} s_0 = \frac{1}{2}$.
- If $\pi \in \mathcal{A}_0^u(G_{2n})$ is regular algebraic, $s_0 = \frac{1}{2}$ is the only possible critical point for which the non-vanishing of the L-value has to be proved. In Namikawa's talk, for the non-triviality of Ash-Ginzburg's p-adic L-function, the non-vanshing $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$ for some $\chi \in \mathbb{X}(p^\infty)$ is needed. (For $\mathrm{GL}(2)$, this is OK by Rohrlich's result.)

3 Known results

- $\mathrm{GL}(1)$: **For Dirichlet characters: [13], [1] etc.,**
  For canonical Hecke characters of CM-fields : [28], [?], [21], [29], [19] etc.

- $\mathrm{GL}(2)$: **[30], [31], [3], [6], [23], [24], [8], [25], [20] etc. Most of these works deal with nonvanishing at $s_0 = \frac{1}{2}$. (We remark that [31] and [6] consider the problem at an arbitrary point s_0).**

- $\mathrm{GL}(n)$ **with** $n > 2$**: [2], [15], [16], [17], [5], [4].**

The non-vanishing result for the central point $s_0 = \frac{1}{2}$ **seems very few. As far as the speaker can find, there are at least two such results:**

- **For** $\pi \in \mathcal{A}_0(G_3)$ **over** $F = \mathbb{Q}$ **by Luo ([15]).**
- **For** $\mathrm{Ad}(\pi)$ $(\pi \in \mathcal{A}_0(G_2))$ **over** $F = \mathbb{Q}$ **by Bump-Friedberg-Hoffstein ([4]).**

There seems no non-vansihing results along $\chi \in \mathbb{X}(p^\infty)$**.**

$\boxed{\textbf{METHODS}}$:

- **The approximate functional equation + a bound of (hyper-)Kloosterman sums (due to Weil and Deligne). [30], [31], [23], [24], [2], [15], [16], [17].**
- **(For quadratic twists)**
  Waldspurger's result on Shimura correspondence + An estimation of Fourier coefficient of half-integral weight forms. ([25])
- **(For anti-cyclotomic twists)**
  Waldspurger type formula identifying the central L-value (degree 4 Euler product) with some toric periods (Waldspurger, S. W. Zhang, Gross) + Equidistribution of Heeger divisors on Shimura varieties. ([35], [36], [20]
- **Metaplectic Eisenstein series and the related multiple Dirichlet series. [3], [6], [4], [5].**

4 Method I

The L-value is captured by the Hecke's zeta integral.

EXAMPLE : $f = \sum_n a(n)q^n \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$.

$$
\begin{aligned}
&\Gamma_{\mathbb{C}}(s)L(s,f)\\
&= \int_0^\infty f(iy)y^s\frac{\mathrm{d}y}{y}\\
&= \int_0^u f(iy)y^s\frac{\mathrm{d}y}{y} + \int_u^\infty f(iy)y^s\frac{\mathrm{d}y}{y}\\
&= (-1)^k\int_{u^{-1}}^\infty f(iy)y^{2k-s}\frac{\mathrm{d}y}{y} + \int_u^\infty f(iy)y^s\frac{\mathrm{d}y}{y}\\
&= (-1)^k\int_{u^{-1}}^\infty \sum_{n=1}^\infty a(n)e^{-2\pi y}\, y^{2k-s}\,\frac{\mathrm{d}y}{y} + \int_u^\infty \sum_{n=1}^\infty a(n)e^{-2\pi y}\, y^s\frac{\mathrm{d}y}{y}.
\end{aligned}
$$

4.1 Rohrlich([30])

- $F = \mathbb{Q}$
- X **: Dirichlet characters with bounded ramification locus.**
- $n = 2$**,** $\pi = \pi_f \in \mathcal{A}_0^u(G_2)$ **with** $f \in S_2^{\mathrm{new}}(\Gamma_0(N), \psi)$
- $s_0 = \frac{1}{2}$

THEOREM 1 (Rohrlich (1984))

- $f \in S_2^{\mathrm{new}}(\Gamma_0(N), \psi)$.
- P : a finite nonempty set of prime numbers not dividing N.

Then there exists $q_0 \in \mathbb{N}$ such that

$$L\left(\frac{1}{2}, \pi_f \otimes \chi\right) \neq 0 \quad \text{for all } \chi \in \mathbb{X}(q) \text{ with } q = \prod_{p \in P} p^{e_p} > q_0.$$

THEOREM 2 (Rohrlich (1984))

- $f \in S_2^{\mathrm{new}}(\Gamma_0(N), \psi)$.
- P **: a finite nonempty set of prime numbers not dividing** N**.**
- $F = \mathbb{Q}(a_f(n) | (n, N) = 1)$ **the Hecke field of** f**.**
- $F(\chi) = F(\chi(n) | n \in \mathbb{N})$**.**

Then,

$$\frac{1}{\#[F(\chi) : F]} \sum_{\sigma \in \mathrm{Hom}_{F-\mathrm{alg}}(F(\chi), \mathbb{C})} L\left(\tfrac{1}{2}, f \otimes \chi^{\sigma}\right) = 1 + o(1), \quad (q \to \infty)$$

for $\chi \in \mathbb{X}(q)$ **with** $q = \prod_{p \in P} p^{e_p} > q_0$

$\boxed{\textbf{Main ingredients}}$ **:**

- **A truncated expression of Hecke's zeta integral.**
- **The two bounds concerning** $\chi_{\mathrm{av}}(n) = [F(\chi) : F]^{-1} \sum_\sigma \chi^\sigma(n)$ **:**

$$\inf\{n \geqslant 2 |\, \chi_{\mathrm{av}}(n) \neq 0\} \gg q^\gamma \quad (\textbf{with some } \gamma > 0),$$

$$\#\{n \in [1,q] |\, \chi_{\mathrm{av}}(n) \neq 0\} = O(1)$$

uniform in χ **and** $q = q_\chi$**.**

- **Weil's bound on the Kloosterman sum.**
- **Shimura's algebraicity is used to ensure** $L\left(\frac{1}{2}, f \otimes \chi\right) \neq 0$ **if and only if** $L\left(\frac{1}{2}, f^\sigma \otimes \chi^\sigma\right) \neq 0$**.**

Remark: **The asymptotic formula implies the non-vanishing of the L-values for** *some* **character-twist of sufficiently large conductor. Shimura's result is used to claim the non-vanishing for** *all* **twists with the same large conductors.**

The following application is given:

THEOREM 3 (Rohrlich (1984))

Let $E/\mathbb{Q}$ be an elliptic curve with CM by the integer ring of an imaginary quadratic field. Let P be a finite set of primes over which E has good reduction. Let $L/\mathbb{Q}$ be the maximal abelian extension unramified outside $P \cup \{\infty\}$. Then $E(L)$ is finitely generated.

$\boxed{\textbf{Novelty}}$ **:**

- **Generic non-vanishing along χ with p-power conductors.**
- **Theorem 1 automoatically implies the** *simultaneous non-vanishing result* **:**
  - $f_i \in S_2^{\mathrm{new}}(N_i, \psi_i)\,(i = 1, \ldots, r)$.
  - P **: a finite set of primes not dividing** $N_1 \ldots N_r$.

  Then there exists q_0 such that

$$L\left(\tfrac{1}{2}, f_i \otimes \chi\right) \neq 0 \quad \textbf{for all } 1 \leqslant i \leqslant r$$

  and for all $\chi \in \mathbb{X}(q)$ with $q = \prod_{p \in P} p^{e_p} > q_0$.

5 Method I'

The L-value is captured by the approximate functional equation.

5.1 Approximate functional equation

Let $\pi \in \mathcal{A}_0^u(G_n)$. **Define** $\{a_\pi(\mathfrak{n}) |$ **integral ideals** $\mathfrak{n} \subset \mathfrak{O}_F\}$ **as**

$$L(s,\pi) = L(s,\pi_\infty) \prod_{v \in \Sigma_{\mathrm{fin}}} L(s,\pi_v) = L(s,\pi_\infty) \sum_{\mathfrak{n}} \frac{a_\pi(\mathfrak{n})}{\mathrm{N}(\mathfrak{n})^s}, \qquad \operatorname{Re} s > 1,$$

where $L(s,\pi_\infty)$ **is the gamma factor. Set** $A(\pi) = D_{F/\mathbb{Q}}^n \mathrm{N}(\mathfrak{q}_\pi)$. **Then**

$$L(s,\pi) = W(\pi)\, A(\pi)^{\frac{1}{2}-s} L(1-s,\pi^\vee).$$

Let $\sigma \in (\frac{1}{2}, 1)$ **and suppose** π_∞ **is tempered for simplicity. For** $y > 0$**, set**

$$\begin{aligned} J(y) &= \tfrac{1}{2\pi i} \int_{(2)} L(s+\sigma, \pi_\infty)\, y^{-s} \frac{\mathrm{d}s}{s}, \\ \tilde{J}(y) &= \tfrac{1}{2\pi i} \int_{(2)} L(1+s-\sigma, \pi_\infty^\vee)\, y^{-s} \frac{\mathrm{d}s}{s}. \end{aligned}$$

For any $u > 0$**, the value** $L(\sigma, \pi)$ **is expressed as the sum of the absolutely convergent series**

$$L(\sigma, \pi) = \sum_{\mathfrak{n}} \frac{a_\pi(\mathfrak{n})}{\mathrm{N}(\mathfrak{n})^\sigma} J(u\mathrm{N}(\mathfrak{n})) + W(\pi) A(\pi)^{\frac{1}{2}-\sigma} \sum_{\mathfrak{n}} \frac{\overline{a_\pi(\mathfrak{n})}}{\mathrm{N}(\mathfrak{n})^{1-\sigma}}\, \tilde{J}\left(\frac{\mathrm{N}(\mathfrak{n})}{A_\pi u}\right),$$

over the set of integral ideals $\mathfrak{n} \subset \mathfrak{O}_F$**.**

Sketch of the proof **: By the contour shift and the functional equation,**

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(2)} L(s+\sigma,\pi)\, u^{-s}\frac{\mathrm{d}s}{s} \\
&= \frac{1}{2\pi i}\int_{(-2)} L(s+\sigma,\pi)\, u^{-s}\frac{\mathrm{d}s}{s} + \mathrm{Res}_{s=0}\left(\frac{L(s+\sigma)u^{-s}}{s}\right) \\
&= \frac{1}{2\pi i}\int_{(-2)} A^{1/2-s-\sigma} L(1-s-\sigma,\pi^{\vee})\, u^{-s}\frac{\mathrm{d}s}{s} + L(\sigma,\pi).
\end{aligned}$$

Substitute the Dirichlet series expansion and exchange the order of integral and summation.

5.2 Rohrlich ([31])

- F : **arbitrary.**
- X : **certain set of** χ **with square free conductor.**
- $n = 1,\ 2$**,** $\pi \in \mathcal{A}_0^u(G_2)$ **no restriction of** ω_π**.**
- $s_0 \in \mathbb{C}$: **arbitrary.**

$\boxed{\textbf{Application}}$: **The condition** $k_v \geqslant 3$ **for all** $v|\infty$ **is removed from the hypothesis of Theorem 4.3 of Shimura's paper ('78 Duke, " The special values of ...")**

THEOREM 1 (Rohrlich1989)

- $\pi \in \mathcal{A}_0^u(G_n)$ **(**$n = 1$ **or** $n = 2$**).**
- $S \subset \Sigma_{\mathrm{fin}}$ **: a finte set.**
- $s_0 \in \mathbb{C}$**.**

There exists infinitely many $\chi \in \mathbb{X}$ unramified over S and square free conductors such that

$$L(s_0, \pi \otimes \chi) \neq 0.$$

$\boxed{\textbf{Asymptotic formula}}$ **:**

- $P(F,S,\pi) := \{p:$ **prime numbers dividing** $D_{F/\mathbb{Q}}\,\mathrm{N}(\mathfrak{q}_\pi)\,\mathrm{N}\prod_{v\in S}(\mathfrak{p}_v)\,\}$**.**
- $\epsilon > 0$ **: small number.**
- $\mathcal{Q}_{S,\pi,\epsilon}$ **: the set of all ideals** $\mathfrak{q} \subset \mathfrak{O}_F$ **s.t.**

(i) $\mathrm{N}(\mathfrak{q})$ **is a product of distinct prime numbers outside** $P(F,S,\pi)$**.**

(ii) $\#\mathbb{X}(\mathfrak{q}) \gg_\epsilon \mathrm{N}(\mathfrak{q})^{1-\epsilon}$**.**

Remark : The technical core of the paper is to show that $\mathcal{Q}_{S,\pi,\epsilon}$ **is an infinite set, which needs an extention of Bombieri-Vinogradov theorem by Murty-Murty.**

• **For $\mathfrak{q} \in \mathcal{Q}_{S,\pi,\epsilon}$ and a prime $r \notin P(F,S,\pi)$ not dividing $q = \mathrm{N}(\mathfrak{q})$, set**

$$X(\mathfrak{q},r) := \{\psi\,(\xi \circ \mathrm{N}_{F/\mathbb{Q}})^j | \,\psi \in \mathbb{X}(\mathfrak{q}),\, 0 \leqslant j \leqslant \tfrac{r-3}{2}\},$$

where ξ is a primitive Dirichlet character of conductor r and of order $\frac{r-1}{2}$.

THEOREM 2 (Rohrlich1989)

Let $\sigma_0 \in \mathbb{R}$ such that $\frac{1}{2} \leqslant \sigma_0 < 1$. Then

$$\frac{1}{\#X(\mathfrak{q},r)} \sum_{\chi \in X(\mathfrak{q},r)} L(\sigma_0, \pi \otimes \chi) = L_\infty(\sigma_0,\pi) + o(1) \quad (\mathrm{N}(\mathfrak{q}) \to \infty)$$

for $\mathfrak{q} \in \mathcal{Q}_{S,\pi,\epsilon}$ and r as above with $\mathrm{N}(\mathfrak{q})^{\epsilon\rho} < r < \mathrm{N}(\mathfrak{q})^{2\epsilon\rho}$ if $\epsilon > 0$ is small enough and $\rho > 0$ is chosen appropriately.

$\boxed{\text{Main ingredients}}$

- **An extention of Bombieri-Vinogradov theorem by Murty-Murty (this is needed to ensure the set $\mathcal{Q}_{\pi,S,\epsilon}$ is infinite.)**
- **The approximate functional equation for $L(s, \pi \otimes \chi)$.**
- **The Rankin-Selberg bound for the average of Fourier coefficients.**
- **Weil's bound of Kloostermann sum (to show the vanishing of the remainder term in the limit.)**

5.3 Barthel and Ramakrishnan ([2])

Methodorogically, this is a direct decendent of Rorlich (1989).

- F **: arbitrary.**
- X**: the same as in Rohrlich (1989).**
- $n \geqslant 3$**,** $\pi \in \mathcal{A}_0^u(G_n)$**.**
- $s_0 \in \mathbb{C}$ **with** $\mathrm{Re}\, s_0 \notin [\frac{1}{n}, 1 - \frac{1}{n}]$
  or $\mathrm{Re}\, s_0 \notin [\frac{2}{n+1}, 1 - \frac{2}{n+2}]$ **if** π **satisfies the Ramanujan conj.**

$\boxed{\text{Motivation}}$: To persue a generalization of the Rohrlich's method to the standard L-function of $\mathrm{GL}(n)$ (and hopefully to other automorphic L-functions like the degree 5 L-functions of $\mathrm{GSp}(4)$).

THEOREM 1 (Barthel-Ramakrishnan (1994)

- $\pi \in \mathcal{A}_0^u(G_n)$ **with** $n \geqslant 3$.
- $S \subset \Sigma_{\mathrm{fin}}$ **: a finte set.**
- $s_0 \in \mathbb{C}$ **s.t.** $\mathrm{Re}\, s_0 \notin [\frac{1}{n}, 1 - \frac{1}{n}]$.

Then there exists infinitly many $\chi \in \mathbb{X}$ **unramified over** S **such that**

$$L(s_0, \pi \otimes \chi) \neq 0.$$

If π **satisfies the Ramanujan conjecture, then the same conclusion holds for** $\mathrm{Re}\, s_0 \notin [\frac{2}{n+1}, 1 - \frac{2}{n+1}]$.

Remark : The theorem has no conclusion for $s_0 = \frac{1}{2}$**.**

THEOREM 2 (Barthel-Ramakrishnan (1994)

Suppose $\mathrm{Re}\, s_0 \notin [\frac{1}{n}, 1-\frac{1}{n}]$ **or** π **is tempered and** $\mathrm{Re}\, s_0 \notin [\frac{2}{n+1}, 1-\frac{2}{n+1}]$**.**

Then

$$\frac{1}{\#X(\mathfrak{q}, r)} \sum_{\chi \in X(\mathfrak{q}, r)} L(\sigma_0, \pi \otimes \chi) = L_\infty(\sigma_0, \pi) + o(1) \quad (\mathrm{N}(\mathfrak{q}) \to \infty)$$

for $\mathfrak{q} \in \mathcal{Q}_{S,\pi,\epsilon}$ **and** r **as above with** $\mathrm{N}(\mathfrak{q})^{\epsilon\rho} < r < \mathrm{N}(\mathfrak{q})^{2\epsilon\rho}$ **if** $\epsilon > 0$ **is small enough and** $\rho > 0$ **is chosen appropriately (to be independent of** $\mathfrak{q}$)**.**

$\boxed{\textbf{Main ingredients}}$:

- **The method is largely an adaptation of the Rohrlich's work [31]. (The use of the special moduli $\mathcal{Q}_{S,\pi,\epsilon}$ is crucial.)**
- **The approximate functional equation for $L(s,\pi\otimes\chi)$.**
- **A bound for the average of Fourier coefficients $\sum_{\mathrm{N}(\mathfrak{n})<X}|a_\pi(\mathfrak{n})|^2 = O(\mathrm{N}(\mathfrak{n})^{1/2-\delta})$ (JPSS theory).**
- **Deligne's bound of the hyper-Kloosterman sum of prime moduli:**

$$K_n(r,q) := \sum_{x_1\ldots x_n\equiv r \pmod q} e\left(\frac{x_1+\cdots+x_n}{q}\right) \leqslant nq^{\frac{n-1}{2}}$$

this is used to show the vanishing of the remainder term in the limit.

5.4 Luo-Rudnick-Sarnak ([16])

- $F = \mathbb{Q}$.
- X : **Dirichlet characters with prime moduli.**
- **For '$\pi \boxtimes \pi^\vee$' with $\Pi \in \mathcal{A}_0^u(G_n)$ unramified at ∞**
- $s_0 \in \mathbb{C}$ **such that** $\mathrm{Re}\, s_0 > 1 - \frac{2}{n^2+1}$.

$\boxed{\textbf{Aim (Motivation)}}$ **: To obtain a bound toward the Selberg conjecture of unramified archimedean parameters of cusp forms on** $\mathrm{GL}(n)$ **(by proving the nonvanishing on** $\mathrm{Re}\, s > 1 - \frac{2}{n^2+1}$ **of the convolution** L**-function** *without gamma factor*.**)**

THEOREM 1 (Luo-Rudnick-Sarnak (1995))

- $F = \mathbb{Q}$.
- $\pi \cong \otimes_v \pi_v \in \mathcal{A}_0^u(G_n)$ **such that** π_∞ **is unramified.**
- $s_0 \in \mathbb{C}$ **such that** $\operatorname{Re} s_0 > 1 - \frac{2}{n^2+1}$.

There exists infinitely many $\chi \in \mathbb{X}(q)$ **(with prime** $q \gg 0$**) such that**

$$L^\infty(s_0, (\pi \otimes \chi) \times \pi^\vee) \neq 0.$$

THEOREM 2 (Luo-Rudnick-Sarnak (1995))

Suppose $\mathrm{Re}\, s_0 \in (0,1)$ **and** $\epsilon > 0$**. Then as** $Q \to \infty$**,**

$$\sum_{\substack{q \text{ is prime} \\ Q<q\leqslant 2Q}} \sum_{\chi\in\mathbb{X}(q)-\{1\}} L^{\infty}(s_0, (\pi\otimes\chi)\times\pi^{\vee}) = \frac{1}{2} \sum_{\substack{q \text{ is prime} \\ Q<q\leqslant 2Q}} q$$

$$+ O_{s_0,\epsilon}(Q^{1+\frac{n^2+1}{2}(1-\mathrm{Re}\, s_0)+\epsilon}).$$

Remark : (1) Since $\sum_{\substack{q \text{ is prime} \\ Q<q\leqslant 2Q}} q \sim \frac{3}{2}Q^2/\log Q$**, the** O**-term has the smaller order than the first term only when** $\mathrm{Re}\, s_0 > 1 - \frac{2}{n^2+1}$**. In this region of** s_0**, the asymptotic formula shows the existence of** χ **such that the twist** $L^{\infty}(s, \Pi\otimes\chi\times\Pi^{\vee})$ **is non zero. Hence the**

gamma factor $L(s, \Pi_\infty \times \Pi_\infty^\vee)$ (independent of χ) is holomorphic on the region $\operatorname{Re} s > 1 - \frac{2}{n^2+1}$.

(2) The result is extended to arbitrary number fields by the same authors ([17]). For that, the use of special moduli constructed by Rohrlich ([31]) is crucial.

$\boxed{\text{Main ingredients}}$:

- **An approximate functional equation for L^∞ (JPSS theory).**
- **Deligne's bound of the hyper-Kloosterman sum .**

THEOREM 3 (Luo-Rudnick-Sarnak (1995))

Let $F = \mathbb{Q}$ **and** $\pi \in \mathcal{A}_0^u(G_n)$ **such that** $\pi_\infty^{O(n)} \neq \{0\}$**.**

Then $\pi_\infty \cong I_{1,\dots,1}(|\,|_{\mathbb{R}}^{s_1}, \dots, |\,|_{\mathbb{R}}^{s_n})$ **with**

$$|\operatorname{Re} s_j| \leqslant \frac{1}{2} - \frac{1}{n^2+1} \quad (1 \leqslant j \leqslant n).$$

($\because$) **From THEOREM 1,** $L(s, \pi_\infty \times \pi_\infty^\vee) = \prod_{j=1}^n \prod_{i=1}^n \Gamma_{\mathbb{R}}(s + s_j + s_i)$

has to be holomorphic on $\operatorname{Re} s > 1 - \frac{2}{n^2+1}$**.**

5.5 Luo ([15])

This refines the result of Barthel-Ramakrishnan by a different method. (Use an averaging similar to that of [16]).

$\boxed{\textbf{Novelty}}$ **: For $n = 3$, the non-vanishing at any point on $\mathrm{Re}\, s = \frac{1}{2}$ is obtained for a twist of any $L(s, \pi)$.**

THEOREM 1 (Luo (2005))

- $F = \mathbb{Q}$
- $\pi \in \mathcal{A}_0^u(G_n)$ **with** $n \geqslant 3$**.**
- $S \subset \Sigma_{\text{fin}}$ **: a finite set over which** π **is unramified.**
- $s_0 \in \mathbb{C}$ **with** $\mathrm{Re}\, s_0 \notin [\frac{2}{n}, 1 - \frac{2}{n}]$**.**

Then there exists infinitly many $\chi \in \mathbb{X}(pr)$ **(with different primes** p, r**) unramified over** S **such that**

$$L(s_0, \pi \otimes \chi) \neq 0$$

(When $n = 3$**,** $[\frac{2}{n}, 1 - \frac{2}{n}]$ **should be understood as** $\varnothing$**.)**

THEOREM 2 (Luo (2005))

Let $n = 3$**. Given a large iteger** $Q > 0$**, set**

$$\mathcal{Q}^S(Q) = \{pr|\ p \in (Q^{\frac{3}{4}}, 2Q^{\frac{3}{4}}],\ r \in (Q^{\frac{1}{4}}, 2Q^{\frac{1}{4}}] \text{ are primes not in } S\},$$

which is a subset of $[Q, 4Q]$**. Let** $\beta \in \mathbb{C}$ **be such that** $\operatorname{Re}\beta \geqslant \frac{1}{2}$**. Then**

$$|\sum_{q \in \mathcal{Q}^S(Q)} \sum_{\chi \in \mathbb{X}(q)} L(\beta, \pi \otimes \chi)| \gg \frac{Q^2}{\log^2 Q} \quad (Q \to \infty).$$

The same asymptotic formula holds for $n > 3$ **with a slightly different choice of the set of moduli** $\mathcal{Q}^S(Q)$ **and for** $\operatorname{Re}\beta \geqslant 1 - \frac{2}{n}$**.**

6 Method III

• **This method relies on the work of Waldspurger on the Shimura correspondence, and works only for quadratic twists of central** L**-value of** $\mathrm{PGL}(2)$**.**

• **The non-vanishing at** $s_0 = \frac{1}{2}$ **of quadratic twist is subtler .**

$\boxed{\textbf{EXAMPLE}}$ **([26]) :** $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ **: Hecke-eigenform.**

• K**: imaginary quadratic field.**

• $\pi = \mathrm{BC}_{K/\mathbb{Q}}(\pi_f)\,(\in \mathcal{A}_0(G_2/K))$

Then $L(\frac{1}{2}, \pi \otimes \chi) = 0$ **for all** $\chi \in \mathbb{X}_F$ **s.t.** $\chi^2 = 1$**.**

6.1 Waldspurger's results

THEOREM ([41], [38] *cf.* [26, Theorem A.2])

- $\pi \cong \otimes_v \pi_v \in \mathcal{A}_0^u(G_2)$ **with** $\omega_\pi = 1$.

Suppose one of the following two conditions is satisfied:

(i) $\exists v \in \Sigma_F$ **s.t.** $\pi_v \in \Pi_{\mathrm{d}}^u(G_2)$.

(ii) $\epsilon(\frac{1}{2}, \pi) = 1$

Then there exists a quadratic $\chi \in \mathbb{X}$ **such that**

$$L\left(\frac{1}{2}, \pi \otimes \chi\right) \neq 0.$$

Conversely, if $L\left(\frac{1}{2}, \pi \otimes \chi\right) \neq 0$ **for some** $\chi \in \mathbb{X}$**, then** π **satisfies one of the above two conditions.**

THEOREM 2 Waldspurger([39]), Ono-Skinner([25])

Let $f \in S_{2k}^{\mathrm{new}}(M,1)$**. Then,**

- $\exists \delta(f) \in \{\pm 1\}$,
- $\exists\, N \in \mathbb{N}$ **s.t.** $4M | N$,
- $\exists\, g = \sum_{n=1}^{\infty} b(n) q^n \in S_{k+\frac{1}{2}}(\Gamma_1(4N))$ **: non-zero**

with the following properties:

(i) For each fundamental discriminant D **s.t.** $\delta(f)D > 0$,

$$b(D_0)^2 = \epsilon_D \frac{L\left(\frac{1}{2}, \pi_f \otimes \chi_D\right) D_0^{k-\frac{1}{2}}}{\Omega_f} \quad \text{if } (D_0, N) = 1$$

and $b(D_0) = 0$ **if** $(D_0, N) > 1$**, where** $D_0 = |D|$ **if** D **is odd and** $D_0 = |D|/4$ **if** D **is even,** Ω_f **is a "period" of** f **and** $\epsilon_D \in \bar{\mathbb{Q}}$**.**

(ii) There exists a number field K such that $b(D_0) \in \mathfrak{O}_K$ for all fundamental discriminant D with $\delta(f)D > 0$.

6.2 Ono-Skinner ([25])

- $F = \mathbb{Q}$
- X: a set of Kroneker characters χ_D of $\mathbb{Q}(\sqrt{D})$.
- $\pi = \pi_f$ with $f \in S_{2k}^{\mathrm{new}}(M, 1)$
- $s_0 = \frac{1}{2}$

Goldfeld conjectured that for $f \in S_{2k}^{\mathrm{new}}(M)$, a positive proportion of $|D| < X$ have the non-vanishing $L(\frac{1}{2}, \pi_f \otimes \chi_D) \neq 0$. They improve the estimate $\gg X^{1-\epsilon}$ for such $|D| < X$ by Perelli-Pomykala(1997) and obtain $\gg X/\log X$.

THEOREM 1 (Ono-Skinner(1998)

- $S = \{p_1, \ldots, p_r\}$ **: prime numbers.**
- $\epsilon = (\epsilon_p)_{p \in S} \in \{\pm 1\}^S$ **: signatures over** S**.**
- $f \in S_{2k}^{\mathrm{new}}(M, 1)$**.**
- $X(\epsilon, S) = \{\chi_D | D$ **: sqr.-free**, $(D, M) = 1,\ \chi_D(p) = \epsilon_p\ (p \in S)\}$**.**

Then, as $X \to \infty$**,**

$$\#\{\chi_D \in X(\epsilon, S) |\, 0 < |D| \leqslant X,\ L\left(\tfrac{1}{2}, \pi_f \otimes \chi_D\right) \neq 0\,\} \gg \frac{X}{\log X}.$$

"Fundamental Lemma" (Ono-Skinner(1998)

- $g^* = \sum_n b^*(n)q^n \in S_{k+\frac{1}{2}}(N^*)$ **s.t.**

(i) $b^*(m) \neq 0$ **for some sqr.-free** $m > 1$ **coprime to** $4N$,

(ii) $b^*(n) \in \mathfrak{O}_K$ **for some number field** $K/\mathbb{Q}$.

- $\delta \in \{\pm 1\}$.
- $v \in \Sigma_K$ **: dividing** 2.
- $s_0 = \{\mathrm{ord}_v(b^*(m)) |\, b^*(m)$ **as in (i)** $\}$.
- $B_{s_0} := \{\delta m |\, m$ **as in (i) with** $s_0 = \mathrm{ord}_v(b^*(m))\}$.
- $P(r) = \{D|$ **product of** r **distinct primes**$\}$.

If $B_{s_0} \cap P(r) \neq \varnothing$**, then**

$$\#\{B_{s_0} \cap P(r) \,|\, |m| < X\} \gg \frac{X}{\log X} (\log \log X)^{r-1}$$

$\boxed{\text{Main Ingredients}}$:

- To show their 'Fundamental Lemma" , they use the ℓ-adic Galois representation attached to a modular form (by Shimura, Deligne and Serre). They use the Chebotatev density theorem to seek large number of primes with some congruence among Fourier coefficients ; the bound $\gg X/\log X$ comes from this.
- Invoke a result by Friedberg-Hoffstein ([6]) to see $b(|D_1|) \neq 0$ for at least one $\chi_{D_1} \in X(\epsilon, S)$.
- Use a quadratic twist of g to obtain g^* with level coprime to D_1 and with Fourier coefficients supported on $m > 1$ such that

$\chi_{\delta m}(p) = \epsilon_p (\forall p \in S)$. **(The Fundamental Lemma should be applied to this g^*).**

7 Method II

This method is based on another work of Waldspurger related to the twisted central L-value

$$L\left(\frac{1}{2}, \mathrm{BC}_{K/F}(\pi) \otimes \chi\right), \qquad \pi \in \mathcal{A}_0(G_2/F).$$

Waldspurger ([40, Théorème 2]

- B/F : **a quaternion algebra.**
- $\pi \in \mathcal{A}_0(G_2)$ **s.t.** $\exists$ **Jacquet-Langlands transfer** $\mathrm{JL}_B(\pi)$ **to** $B^\times(\mathbb{A})$**.**
- $K \hookrightarrow B$: **a quadratic extension of** F **embeded into** B
- χ : **a Hecke character of** $\mathbb{A}_K^\times$ **s.t.** $\chi|\mathbb{A}_F^\times = \omega_\pi$**.**

Then

$$\ell_{K,\chi}(\psi) := \int_{\mathbb{A}_F K^\times \backslash \mathbb{A}_K^\times} \psi(t)\chi(t)^{-1}\,\mathrm{d}t \neq 0 \quad (\exists \psi \in \mathrm{JL}_B(\pi))$$

if and only if

(i) $\dim_{\mathbb{C}} \mathrm{Hom}_{B_v^\times}\left(\sigma'_v, \mathrm{Ind}_{K_v^\times}^{B_v^\times}(\chi_v)\right) \neq 0\,(\forall v \in \Sigma_F)$**,**

(ii) $L\left(\frac{1}{2}, \mathrm{BC}_{K/F}(\pi) \otimes \chi\right) \neq 0$

$\boxed{L\text{-value formula}}$:

(Under a correct condition of local ϵ-factors and and local invariants of B), the Waldspurger's formula and its refinements ([7], [42], [43] for CM case, [27] for real quadratic field $K/\mathbb{Q}$) take the form

$$L\left(\tfrac{1}{2}, \mathrm{BC}_{K/F}(\pi) \otimes \chi\right) = C \, \frac{\|f\|^2}{\|\psi_f\|^2} \left|\ell_{K,\chi}(\psi_f)\right|^2$$

with a positive constants C, where $(f, \psi_f) \in \pi \times \mathrm{JL}_B(\pi)$ is a certain pair of test vectors.

7.1 Vatsal ([35])

- $K = \mathbb{Q}(\sqrt{-D})$ **: imaginary quadratic field of discriminant** $-D$
- X **: anti-cyclotomic char's of** p**-power conductor (**$(p, 2D) = 1$**).**
- $\pi = \mathrm{BC}_{K/\mathbb{Q}}(\pi_g)$ **, where** $g \in S_2^{\mathrm{new}}(N)$ **with** $(p, N) = 1$**.**
- $s_0 = \frac{1}{2}$**.**

Aim (Motivation) : (non CM case for) Mazur's conjecture (on the finite generation of the anti-cyclotomic part of $E(K)$ **for an elliptic curve** $E/\mathbb{Q}$ **with** K **varying along a tower of ray class fields of** p**-power conductor over an imaginary quadratic field.)**

Theorem 1 (Vatsal([35])

- N^+ : an integer.
- N^- : square free prime to N^+, product of odd number of primes inert in K.
- p : a prime s.t. $(p, 2DN^+N^-) = 1$.
- $\mathbb{X}^-(p^\infty)$: the set of all those anti-cyclotomic finite order characters of K with p-power conductor.
- $g \in S_2^{\mathrm{new}}(N^+N^-)$: Hecke-eigen form.

Suppose (i) p is ordinary for g or (ii) p does not divide $h_K = \#\mathrm{Cl}_K$.

Then $L\left(\frac{1}{2}, \mathrm{BC}_{K/\mathbb{Q}}(g) \otimes \chi\right) \neq 0 \quad$ except for finitely many $\chi \in \mathbb{X}^-(p^\infty)$.

For simplicity, we suppose $h_K = 1$ (thus D is a prime) here to simplify the statement of the next theorem.

Theorem 2 (Vatsal([35])

For $\chi \in X$, let $\chi = \chi_t \chi_w$ be the decomposition to the tame part χ_t (of order prime to p) and the wide part χ_w (of p-power order). Set $\mathbb{X}^-(p^n)[\chi_t]$ be the set of $\chi \in \mathbb{X}^-(p^\infty)$ of conductor p^n with the fixed tame part χ_t. Then

$$\frac{1}{p^{n-1}} \sum_{\chi \in \mathbb{X}^-(p^n)[\chi_t]} \frac{L\left(\frac{1}{2}, \mathrm{BC}_{F/\mathbb{Q}}(g) \otimes \chi\right)}{\Omega_g} = \left(1 + \chi_t(\mathrm{Frob}(\mathfrak{D}))\frac{a_g(D)}{D+1}\right) \|\psi_g\|^2 +$$

where ψ_g is a normalized Jacquet-Langlands companion on $\mathrm{Cl}(B)$ of g.

Main ingredients :

- B **is the definite quaternion over** $\mathbb{Q}$ **and consider Eichler orders** $R \subset B$ **of level** N^+. **The JL companion** ψ_g **of** g **lives on the finite set** $\mathrm{Cl}(B) = B^\times \backslash \hat{B}^\times / \hat{R}^\times$.

- **Gross's formula** **([7]) to express the** L**-values by the square value of the function** ψ_g **at a** χ**-twisted sum of "Heeger points"** $P = (f, R)$ **over** $\mathrm{Cl}(B)$.

- **Equidistribution of Heeger points of** p**-power conductor** **(this is used to compute the main term coming from the trivial character of** K**).**

- **Ratner's theorem on unipotent flows on (p-adic) Lie groups (this is used to show the vanishing of the remainder terms in the limit.)**

- **Shimura's reciprocity of $L(\frac{1}{2}, g \times \theta_\chi)$ under Galois twists is used to extend the non-vanishing to all relevant characters but for finitely many exception (an idea introduced by Rohrlich [30]).**

7.2 Michel-Venkatesh ([20])

This is largely an expository article. They explain several ways to obtain a quantitative non-vanishing result for the "non-abelian" twists

$$L(\tfrac{1}{2}, f \times \theta_\chi), \qquad \chi \in \widehat{\mathrm{Cl}_K}$$

of $L(s,f)$ **by the theta series** θ_χ **induced from a class group character** χ **of an imaginary quadratic field** $\mathbb{Q}(\sqrt{-D})$ **as** D **grows, where** $\theta_\chi(z) = \sum_{(0)\neq\mathfrak{a}\subset\mathfrak{O}_K} \chi(\mathfrak{a}) q^{\mathrm{N}(\mathfrak{a})} \in S_1(\Gamma_0(D), \chi_D)$**.**

(Strictly speaking, this is out of the framework of this talk.)

Theorem (Michel-Venkatesh (2005))

- $f \in S_2^{\mathrm{new}}(q,1)$ **with prime level** q.
- $K = \mathbb{Q}(\sqrt{-D})$ **: an imaginary quad. field of disc.** $-D$**. s.t.** q **is inert in** K**.**

Then

$$\sum_{\chi \in \widehat{\mathrm{Cl}(D)}} L\left(\tfrac{1}{2}, f \otimes \theta_\chi\right) \gg_{\epsilon,f} D^{\frac{1}{2}-\epsilon}, \quad D \to \infty.$$

For any $0 < \delta < 1/2700$**,**

$$\#\{\chi \in \widehat{\mathrm{Cl}_K} \,|\, L\left(\tfrac{1}{2}, f \times \theta_\chi\right) \neq 0\} \gg_{\delta,f} D^{\delta}, \quad (D \to \infty).$$

$\boxed{\text{Skech of the proof}}$:

- B: the definite quoternion ramified at ∞q.

- From Gross's formula

$$\sum_{\chi \in \widehat{\mathrm{Cl}(D)}} L\left(\tfrac{1}{2}, f \otimes \theta_\chi\right) = \frac{h_K \|f\|^2}{4\sqrt{D}} \sum_{\sigma \in \mathrm{Cl}(D)} |\psi_f(x_\sigma)|^2$$

where $\psi_f : \mathrm{Cl}(B) \to \mathbb{C}$ is an appropriate JL companion of f such that $\|\psi_f\|^2 = 1$.

- Use the fact (Iwaniec) that the points $\{x_\sigma | \sigma \in \mathrm{Cl}(D)\}$ become equidistributed on $\mathrm{Cl}(B)$ to conclude

$$h_K^{-1} \sum_\sigma |\psi_f(x_\sigma)|^2 = (1 + o(1))\|\psi_f\|^2, \quad D \to \infty.$$

- **Use Siegel's bound of class numbers** $h_K \gg_\epsilon D^{1/2-\epsilon}$ **to obtain the lower bound from the asymptotic formula:**

$$\sum_{\chi \in \widehat{\mathrm{Cl}(D)}} L\left(\tfrac{1}{2}, f \otimes \theta_\chi\right) \sim \frac{h_K^2 \|f\|^2}{4\sqrt{D}} \gg_\epsilon \frac{D^{1-\epsilon}}{D^{1/2}} = D^{1/2-2\epsilon}$$

- **Invoke the subconvex bound** $L(1/2, f \times \theta_\chi) \ll_f D^{1/2-\delta})$ **for any** $0 < \delta < 1/1100$ **(P. Michel).**

8 Method IV

Based on the study of the double Dirichlet series

$$\sum \frac{L(w, \pi \otimes \chi_d)}{\mathrm{N}(d)^s}$$

(χ_d **: quadratic Dirichlet character assocoated to** $F(\sqrt{m})/F$**)**

8.1 Friedberg and Hoffstein ([6])

- F **: arbitrary**
- X **: A set of quadratic characters**
- $n = 2$, $\pi \in \mathcal{A}_0(G_2)$ **(not nec. unitary)**
- $s_0 = \frac{1}{2}$

THEOREM (Friedberg and Hoffstein ([6]))

- S : **a finite set of places**
- $\xi \in \mathbb{X}$: **a fixed quadratic character**
- $\Psi(S;\xi) = \{\chi \in \mathbb{X} |$ **quadratic,** $\xi_v = \chi_v \, (\forall v \in S) \, \}$
- $\pi \in \mathcal{A}_0(G_2)$ **(not nec. unitary)**

If (i) π **is not selfdual or (ii)** π **is self-dual, and suppose** $\epsilon(\frac{1}{2}, \pi \otimes \chi) = 1$ **for some** $\chi \in \Psi(S;\xi)$**, then there exists infinitely many** $\chi \in \Psi(S;\xi)$ **such that**

$$L(\tfrac{1}{2}, \pi \otimes \chi) \neq 0.$$

8.2 Other works

• **Bump, D, Friedberg, S., Hoffstein, J.,** *Nonvanishing theorems for L-functions of modular forms and their derivatives*, **Invent. Math.** 102 **(1990), 543-618.**

• **Chinta, G,. Friedberg, S., Hoffstein, J.,** *Asymptotics for sums of twisted L-functions and applications*, **In Automorphic representations, L-functions and Applications: progress and perspective, 75-94, Ohaio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.**

• **Bump, D, Friedberg, S., Hoffstein, J.,** *Sums of twisted* GL(3) *automorphic L-functions*, **Contributions to Automorphic forms , Geometry and Number Theory, Chapter 7, The Johns Hopkins University Press, Baltimore and London, 2004.**

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