GALOIS REPRESENTATION ASSOCIATED TO ELLIPTIC MODULAR FORMS

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Contents

1. Statement of the main theorem

Definition 1.1. Let f be a normalized newform of weight $k \geq 2$. Let us consier the following conditions for a given l-adic representation $\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V_f) \cong GL_2(\overline{\mathbb{Q}}_l)$:
(L) The L-function $L(V_f, s)$ coincides with $L(f, s)$ for $\mathrm{Re}(s) \gg 0$.
(Ir) ρ is irreducible as a representation of $G_{\mathbb{Q}}$.
(Geom) The representation has an giometric origin (for a pure motive of weight $k-1$).

Theorem A (Deligne, Ribet)
Let f be a normalized newform of weight $k \geq 2$, level $\Gamma_1(M)$. For every prime l, we have a continuous Galois representation V_f of rank 2 over $\overline{\mathbb{Q}}_l$ which satisfies the conditions (L), (Ir), (Geom).

The construction by Deligne shows a stronger statement that the L-function $L(V_f, s)$ is convergent for $\mathrm{Re}(s) > \frac{k+1}{2}$. His stronger statement implies the following corollary.

Corollary A (Deligne) Let f be a normalized newform of weight $k \geq 2$, level $\Gamma_1(M)$. Then the generalized Ramanujan conjecture for f is true. That is $|a_p(f)| \geq 2p^{\frac{k-1}{2}}$ for every $p \nmid M$.

Proof of Theorem A $\Rightarrow$ Corollary A
(Geom) + Weil conjecture (theorem by Deligne).

Remark

(1) Let f be a normalized newform of weight $k \geq 2$, level $\Gamma_1(M)$. The construction for V_f satisfying only (L) is know by Shimura. (l-adic approximation method) Note that Shimura's proof of Theorem A does not imply Corollary A.

(2) Under (L) and (Ir), the representation is unique modulo isomorphism (Chebotarev density theorem)

2. Construction of the modular Galois representation

Let us consider:
$Y_1(M)_{\mathbb{Q}}$: affine modular curve of level $\Gamma_1(M)$.
$X_1(M)_{\mathbb{Q}}$: proper modular curve of level $\Gamma_1(M)$.
$\mathfrak{Y}_1(M)$: model of $Y_1(M)_{\mathbb{Q}}$ over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{M}])$
$\mathfrak{X}_1(M)$: model of $X_1(M)_{\mathbb{Q}}$ over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{M}])$
All these are obtained as a moduli of elliptic curves or generalized elliptic curves (cf. a book of Katz-Mazur).
Under the geometric situation:

$$\begin{array}{c}\mathcal{E}\\ \downarrow \pi\\ \mathfrak{Y}_1(M),\end{array}$$

we introduce the following sheaf:

$$\mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l) = j_*\left(\varprojlim_n \mathrm{Sym}^{k-2}(R^1\pi_*(\mathbb{Z}/l^n\mathbb{Z}))\right)\otimes_{\mathbb{Z}_l}\overline{\mathbb{Q}}_l,$$

where j is the open immersion $\mathfrak{Y}_1(M)\hookrightarrow\mathfrak{X}_1(M)$. For any closed geometric point $\overline{\kappa}\cong\mathrm{Spec}(\overline{\mathbb{F}}_p)$ of $\mathfrak{X}_1(M)$ of characteristic $p\nmid Ml$, $\mathcal{L}(\overline{\mathbb{Q}}_l)|_{\overline{\kappa}}$ is pure of weight $k-2$. Let us consider the etale cohomology

$$H^1_{\mathrm{et}}(X_1(M)_{\overline{\mathbb{Q}}},\mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))$$

on which the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts compatibly with the Hecke action and the representation.

We put

$$\begin{aligned}V_f &:= H^1_{\mathrm{et}}(X_1(M)_{\overline{\mathbb{Q}}},\mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))[\pi_f]\\ &:= \bigcap_{p\nmid Ml}\mathrm{Ker}[H^1_{\mathrm{et}}(X_1(M)_{\overline{\mathbb{Q}}},\mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))\xrightarrow{T_p-a_p(f)}H^1_{\mathrm{et}}(X_1(M)_{\overline{\mathbb{Q}}},\mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))]\end{aligned}$$

In order to show Theorem A with stronger version, we need to show the following three important statements.
(I) The f-component $V_f := H^1_{\mathrm{et}}(X_1(M)_{\overline{\mathbb{Q}}},\mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))^{\mathrm{new}}[\pi_f]$ on which the Hecke algebra is of dimension two over $\overline{\mathbb{Q}}_l$.
(II) The action of $G_{\mathbb{Q}}$ is unramified outside $Ml\infty$. For a prime $p\nmid Ml$, the (conjugation class of) geometric Frobenius $\mathrm{Frob}_p\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acting on V_f has absolute values which are algebraic numbers whose absolute values are $p^{\frac{k-1}{2}}$ (independently of $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$).
(III) For a prime $p\nmid Ml$, the trace of $\mathrm{Frob}_p\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acting on V_f coincides with p-th

Fourier coefficient $a_p(f)$.

As for (I), We recall the following isomorphism:

$$\begin{aligned} H^1_{\mathrm{et}}(X_1(M)_{\overline{\mathbb{Q}}}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))^{\mathrm{new}} &\cong H^1_{\mathrm{et}}(X_1(M)_{\mathbb{C}}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))^{\mathrm{new}} \\ &\quad \text{(induced by a fixed embedding } \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}) \\ &\cong H^1_{\mathrm{Betti}}(X_1(M)_{\mathbb{C}}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))^{\mathrm{new}} \\ &\quad \text{(Comparison between Etale and Betti)} \\ &\cong H^1_{\mathrm{Betti}}(X_1(M)_{\mathbb{C}}, \mathcal{L}^{(k-2)}(\mathbb{C}))^{\mathrm{new}} \\ &\quad \text{(induced by a fixed isomorphism } \overline{\mathbb{Q}}_l \cong \mathbb{C}) \\ &\cong S_k(\Gamma_1(M); \mathbb{C})^{\mathrm{new}} \oplus \overline{S_k(\Gamma_1(M); \mathbb{C})^{\mathrm{new}}} \\ &\quad \text{(Eichler-Shimura isomorphism)} \end{aligned}$$

On the other hand, we define the Hecke algebra $\mathbb{T}_M$ to be the image of the abstract Hecke algebra $\mathbb{Q}[\{T_n\}_{n\nmid M}]$ in $\mathrm{End}_{\overline{\mathbb{Q}}_l}(H^1_{\mathrm{et}}(X_1(M)_{\overline{\mathbb{Q}}}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))^{\mathrm{new}})$. Taking the f-part $[\pi_f]$ of the $\mathbb{T}_M$-action on the above isomorphism, we see that $V_f = H^1_{\mathrm{et}}(X_1(M)_{\overline{\mathbb{Q}}}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))^{\mathrm{new}}[\pi_f]$ is of dimension two over $\overline{\mathbb{Q}}_l$. This shows (I).

In order to prove (II), we need to consider:
$\mathfrak{Y}_1(M)_{\mathbb{F}_p}$: mod p reduction of the model $\mathfrak{Y}_1(M)$ over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{M}])$
$\mathfrak{X}_1(M)_{\mathbb{F}_p}$: mod p reduction of the model of $\mathfrak{X}_1(M)$ over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{M}])$.

For the following similar geometric situation as in the previous one:

$$\begin{array}{c} \mathcal{E}_{\mathbb{F}_p} \\ \downarrow \pi_{\mathbb{F}_p} \\ \mathfrak{Y}_1(M)_{\mathbb{F}_p}, \end{array}$$

we introduce the following sheaf:

$$\mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l) = j_* \left(\varprojlim_n \mathrm{Sym}^{k-2}(R^1(\pi_{\mathbb{F}_p})_*(\mathbb{Z}/l^n\mathbb{Z})) \right) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l,$$

where j is the open immersion $\mathfrak{Y}_1(M)_{\mathbb{F}_p} \hookrightarrow \mathfrak{X}_1(M)_{\mathbb{F}_p}$.

Let us consider a proper smooth morphism $q : \mathfrak{X}_1(M) \longrightarrow \mathbb{Z}[\frac{1}{M}]$.

Theorem (Proper base change theorem)
For each $p \nmid M$,

$$(R^1 q_* \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))_{\overline{\mathbb{F}}_p} \cong H^1_{\mathrm{et}}(\mathfrak{X}_1(M)_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l)).$$

Similarly, we have:

$$\begin{aligned}(R^1 q_* \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))_{\overline{\mathbb{Q}}} &\cong H^1_{\text{et}}(\mathfrak{X}_1(M)_{\overline{\mathbb{Q}}}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l)) \\ &\cong H^1_{\text{et}}(X_1(M)_{\overline{\mathbb{Q}}}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))\end{aligned}$$

Since q is smooth, the sheaf $(R^1 q_* \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))$ over $\mathbb{Z}[\frac{1}{Ml}]$ is constant. Hence, for $p \nmid Ml$, we have

$$(R^1 q_* \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))_{\overline{\mathbb{Q}}} \cong (R^1 q_* \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))_{\overline{\mathbb{F}}_p}$$

as a module over $\pi_1(\mathbb{Z}[\frac{1}{Ml}], *) \cong \mathrm{Gal}(\mathbb{Q}_{\Sigma_{Ml}}/\mathbb{Q})$ where $\mathbb{Q}_{\Sigma_{Ml}}$ is the maximal Galois extension unramified outside Ml. Further, since the sheaf $\mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l)$ is pure of weight $k-2$ on $\mathfrak{X}_1(M)_{\overline{\mathbb{F}}_p}$, we see that the eigen values of Frob_p on $H^1_{\text{et}}(\mathfrak{X}_1(M)_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l)) \cong H^1_{\text{et}}(\mathfrak{X}_1(M)_{\overline{\mathbb{Q}}}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))$ are algebraic numbers whose complex absolute values are $p^{\frac{k-1}{2}}$. This shows (II).

In order to show (III), we assume that the weight k is equal to 2 where $\mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l)$ is a constant sheaf. By using congruence (weight 2 $\longleftrightarrow$ weight k) modulo power of l, we can justify "the reduction to weight 2".

Let Φ_p : be the endomorphism on by taking "the p-th power of coordinates" on $(\mathfrak{X}_1(M)_{\mathbb{F}_p})_{\overline{\mathbb{F}}_p}$.

The endomorphism induced on $H^1_{\text{et}}((\mathfrak{X}_1(M)_{\mathbb{F}_p})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(k-2)}(\overline{\mathbb{Q}}_l))^{\text{new}}[\pi_f]$ by Φ_p is also denoted by the same symbol Φ_p.

It is a fundamental property of the etale cohomology that Φ_p is equivalent to the action of the geometric Frobenius element Frob_p. We compare the action of Φ_p with Hecke correspondence.

We consider the following elements in $\mathrm{Corr}(\mathfrak{X}_1(M), \mathfrak{X}_1(M))$

(1) T_p: Hecke correspondence at p

(2) $\langle p \rangle$: diamond operator at p

Recall that $\mathfrak{X}_1(M)$ parametrizes isomorphism classes of (E, e) where E is an elliptic curve and e is an element of order M.

It is checked that, over $\mathbb{Q}$, we have:

(1) T_p sends (E, e) to $\sum_C (E/C, e + C)$ where $C \subset E$ runs through cyclic subgroups of order p such that $C \cap \langle e \rangle = \{0\}$.

(2) $\langle p \rangle$ sends (E, e) to (E, pe).

We denote by $\widetilde{T_p}, \widetilde{\langle p \rangle} \in \mathrm{Corr}(\mathfrak{X}_1(M)_{\mathbb{F}_p}, \mathfrak{X}_1(M)_{\mathbb{F}_p})$ the reduction modulo p of T_p and $\langle p \rangle$.

Note that, if E has ordinary reduction at p, we have

$$\widetilde{(E/C, e + C)} = (\widetilde{E}^{\sigma_p}, \widetilde{e}^{\sigma_p}) \qquad \text{if } C \text{ is the unique canonical subgroup}$$

$$\widetilde{(E/C, e + C)} = (\widetilde{E}^{\sigma_p^{-1}}, p\widetilde{e}^{\sigma_p^{-1}}) \qquad \text{if } C \text{ is one of the } p \text{ non-canonical subgroup}$$

Note that a correspondence induces an endomorphism on the cohomology. Thus we obtain:

Theorem(Congruence relation)
As an endomorphism on $H^1_{\text{et}}(\mathfrak{X}_1(M)_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_l)$, we have:

$$\Phi_p + p\widetilde{\langle p\rangle}(\Phi_p)^{-1} = \widetilde{T_p}.$$

From this relation we have:

$$\mathrm{tr}(\mathrm{Frob}_p) = \mathrm{tr}(\Phi_p) = \mathrm{tr}\widetilde{T_p} = a_p(f)$$

on $H^1_{\text{et}}(\mathfrak{X}_1(M)_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_l)^{\text{new}}[\pi_f]$.

3. Proof of the irreducibility

We will prove the (Ir) basically following the proof by Ribet. Suppose that the Galois representation ρ_f (on V_f) is not irreducible. Then we have

$$0 \longrightarrow V_1 \longrightarrow V_f \longrightarrow V_2 \longrightarrow 0,$$

where V_1 and V_2 are Galois representation of $G_{\mathbb{Q}}$ which are of dimension 1 over $\overline{\mathbb{Q}}_l$. The images of $G_{\mathbb{Q}}$ in $\mathrm{Aut}(V_1)$ and $\mathrm{Aut}(V_2)$ are l-adic Lie groups. Hence, $G_{\mathbb{Q}}$ acts on each V_i through the character $\chi_{cyc}^{a_i}\epsilon_i$ with $a_i \in \mathrm{Z_p}$ and ϵ_i characters of finite order.

(1) Since the Galois representation V_f restricted to the decomposition group at l is Hodge-Tate at l, a_1 and a_2 are integer.

(2) Since $\det V_f$ is χ_{cyc}^{1-k}, $a_1 + a_2 = 1 - k$.

(3) By Weil conjecture, for every prime $p \nmid Nl$, we have

$$|a_p(f)| = |p^{-a_1}\epsilon_1(p) + p^{-a_2}\epsilon_2(p)| \leq 2p^{\frac{k-1}{2}}.$$

If we choose a prime p satisfying $\epsilon_1(p) = \epsilon_2(p)$, we have

$$|p^{-a_1} + p^{-a_2}| \leq 2p^{\frac{k-1}{2}}.$$

If $a_1 \neq a_2$, we have $p^{-a_1} + p^{-a_2} > 2\sqrt{p^{-(a_1+a_2)}} = 2p^{\frac{k-1}{2}}$. Thus a_1 must be equal to a_2. Thus, we deduce that

(a) k is odd.

(b) $a_1 = a_2 = \frac{k-1}{2}$.

(4) Since k is odd, $\epsilon_1\epsilon_2$ is an odd character. Hence, $\epsilon_1\epsilon_2^{-1} = (\epsilon_1\epsilon_2)(\epsilon_2^{-1})^2$ is also an odd character. Especially, we show that $\epsilon_1\epsilon_2^{-1} \neq \mathbf{1}$.

(5) We have

$$\sum_{p\nmid Ml} |a_p(f)|^2 p^{-s} = \sum_{p\nmid Ml} |1 - \epsilon_1\epsilon_2^{-1}(p)|^2 p^{k-1-s}$$

If we write $\epsilon_1\epsilon_2^{-1}(p) = e^{\frac{2n_p\pi i}{c}}$ with c the order of $\epsilon_1\epsilon_2^{-1}$ for some $n_p \in \mathbb{Z}/c\mathbb{Z}$, we have

$$|1 - \epsilon_1\epsilon_2^{-1}(p)|^2 = (1 - e^{\frac{2n_p\pi i}{c}})(e^{-\frac{2n_p\pi i}{c}}) = 2 - 2\cos\frac{2n_p\pi i}{c}.$$

Recall that

$$\sum_{p\nmid Ml} p^{k-1-s} = -\log(s-k) + O(1) \quad (\text{when } s \longrightarrow k+)$$

Since $c > 1$ and n_p is equidistributed modulo c, the contribution of $\cos \frac{2n_p\pi i}{c}$ for moving p vanishes. Hence, we have

$$\sum_{p \nmid Ml} |a_p(f)|^2 p^{-s} = -2\log(s-k) + O(1) \quad (\text{when } s \longrightarrow k+)$$

(6) On the other hand, Rankin studies the L-function $L(g \otimes g, s)$ and shows that it has a simple pole of residue 1 at $s = k$ for every cusp form g of weight $k \geq 2$. This implies

$$\sum_{p \nmid Ml} |a_p(f)|^2 p^{-s} = -\log(s-k) + O(1) \quad (\text{when } s \longrightarrow k+)$$

This implies the contradiction. Hence the Galois representation is irreducible.

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