# ジーゲルモジュラー形式入門

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## 1. Siegel upper half space $\mathfrak{H}_n$

$$G = Sp_{2n}(\mathbb{R})$$
$$= \left\{ g \in GL_{2n}(\mathbb{R}) \,\middle|\, {}^t g J_n g = J_n = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \right\}$$

$$GSp_{2n}(\mathbb{R})$$
$$= \left\{ g \in GL_{2n}(\mathbb{R}) \,\middle|\, {}^t g J_n g = \nu(g) J_n \quad \nu(g) \in \mathbb{C}^\times \right\}$$

$$\mathfrak{H}_n = \left\{ Z = X + \sqrt{-1}\, Y \in M_n(\mathbb{C}) \,\middle|\, {}^t Z = Z,\ Y > 0 \text{ pos. definite} \right\}$$

$$G \curvearrowright \mathfrak{H}_n$$
$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G, \quad Z \in \mathfrak{H}_n$$

Claim
$$CZ + D \in GL_n(\mathbb{C})$$

$$g\langle Z \rangle := (AZ+B)(CZ+D)^{-1} \in \mathfrak{H}_n$$

<u>Claim</u>
- The action is transitive
- $K = \mathrm{Stab}(\underbrace{\sqrt{-1}\,\mathbb{1}_n}_{= Z_0}) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \right\} \cong U(n)$, $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}\, B$

Then $\mathfrak{H}_n \cong G/K$
$$g\langle Z_0 \rangle \longleftarrow g$$

## 2. Definition of Siegel modular form

$$\Gamma_n = Sp_{2n}(\mathbb{Z}) = G \cap M_{2n}(\mathbb{Z})$$

$J : G \times \mathfrak{H}_n \longrightarrow GL_n(\mathbb{C})$ automorphic factor

$$\left(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z\right) \longmapsto CZ + D$$

$\rho$: irreducible rational finite dim rep. of $GL_n(\mathbb{C})$ on $V$

<u>Def</u> $F : \mathfrak{H}_n \xrightarrow{C^\infty} V$ is holomorphic Siegel modular form of wt $\rho$ w.r.t $\Gamma_n$

$\Leftrightarrow$ ① $F$ is hol. function on $\mathfrak{H}_n$

② $F(\gamma\langle Z\rangle) = \rho(J(\gamma, Z))F(Z)$ $\quad \forall \gamma \in \Gamma_n \quad \forall Z \in \mathfrak{H}_n$

[③ $(n=1)$ $F$ is holomorphic at cusp]

$M_\rho(\Gamma_n) :=$ the space of hol. Siegel modular form of wt $\rho$ w.r.t. $\Gamma_n$.

$\cdot\ \rho = \det^k$ $(V = \mathbb{C})$ $\quad$ wt $k$ $\quad M_k(\Gamma_n)$

$\cdot\ \rho \longleftrightarrow (k_1, \cdots, k_n) \in \mathbb{Z}^n \quad k_1 \geqq k_2 \geqq \cdots \geqq k_n$

<u>$n=2$</u> $\quad \rho_{k_1, k_2} = \mathrm{Sym}^{k_1 - k_2} \otimes \det^{k_2}$, $\quad V = \mathbb{C}^{k_1 - k_2 + 1}$

lifting to the group $G$

$F \in M_\rho(\Gamma_n)$

$\varphi(g) = \varphi_F(g) = \rho(J(g, Z_0))^{-1} F(g\langle Z_0\rangle)$ $\quad g \in G$

② $\Leftrightarrow$ $\varphi(\gamma g k) = \rho(k)^{-1}\varphi(g)$

$\forall \gamma \in \Gamma_n \ \forall g \in G, \ \forall k \in K$ $\qquad X_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm\sqrt{-1} \\ \pm\sqrt{-1} & -1 \end{pmatrix}$

① $\Leftrightarrow$ $X * \varphi = 0 \quad (X \in \mathfrak{p}_-)$

$\mathfrak{g} = \mathrm{Lie}(G) = \{X \in M_{2n}(\mathbb{R}) \mid {}^tX J_n + J_n X = 0\}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

$$\mathfrak{k} = \mathrm{Lie}(K) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M_{2n}(\mathbb{R}) \;\middle|\; \begin{array}{l} {}^tA = -A \\ {}^tB = B \end{array} \right\}$$

$$\mathfrak{p} = \left\{ \qquad\qquad \;\middle|\; {}^tA = A,\ {}^tB = B \right\}$$

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

$$\mathfrak{p}_\pm = \left\{ \begin{pmatrix} A & \pm\sqrt{-1}A \\ \pm\sqrt{-1}A & -A \end{pmatrix} \;\middle|\; {}^tA = A \right\}$$

$$(X * \varphi)(g) = \left.\frac{d}{dt}\right|_{t=0} \varphi(g \exp tX)$$

## 3. Fourier expansion

$n = 1$

$$F(z) = F(z+1) = \sum_{n\in\mathbb{Z}} a_n(y)\, e^{2\pi i n x}$$

$$\overset{\text{hol}}{=} \underset{\substack{\uparrow \\ \text{hol. at cusp}}}{\sum_{n \geq 0}} a_n \cdot \underbrace{e^{-2\pi n y}\, e^{2\pi i n x}}_{e^{2\pi i n z}}$$

<u>Thm</u> $F \in M_\rho(\Gamma_n)$

$$F(Z) = \sum a_F(T) \exp\left[2\pi\sqrt{-1}\, \mathrm{Tr}(TZ)\right]$$

$T$: semi-integral sym.

$\left(\text{diag} \in \mathbb{Z},\ \text{other} \in \frac{\mathbb{Z}}{2}\right)$

$\boxed{T \geq 0}$ Koecher principle

①, ②

$F$ : non-holomorphic

$\left( \det T = 0 \qquad \overset{n=2}{\text{Whittaker function}} \right)$

$$a_F(T) = a_F(VT\,{}^tV) \qquad V \in SL_n(\mathbb{Z})$$

Siegel operator $\Phi$

$F \in M_\rho(\Gamma_n)$

$$(\Phi F)(Z') := \lim_{\lambda \to \infty} F\begin{pmatrix} Z' & 0 \\ 0 & \sqrt{-1}\lambda \end{pmatrix} \qquad Z' \in \mathfrak{H}_{n-1}$$

$\Phi F \in M_\rho(\Gamma_{n-1})$ $\quad k_1, \cdots, k_{n-1}$

$F$ is a cuspform $\underset{\text{def}}{\Longleftrightarrow}$ $\Phi F = 0$

$\Longleftrightarrow \left( a_F(T) \neq 0 \Rightarrow T > 0 \right)$

$$S_\rho(\Gamma_n) = \{ F \in M_\rho(\Gamma_n) \mid F \text{ is a cusp form} \}$$

<u>Peterson inner product</u>

$f, g \in S_\rho(\Gamma_n)$

for $GL_2$

$\left( \int_{\Gamma_n \backslash \mathfrak{H}_n} f(z)\overline{g(z)} (\det Y)^k d^*Z \right)$

$$(f, g) = \int_{\Gamma_n \backslash \mathfrak{H}_n} \langle \rho(Y^{1/2}) f(Z), \rho(Y^{1/2}) g(Z) \rangle$$

$f, g \in S_k(\Gamma_n)$

$f$ と $g$ を $G$ 上に lift したとき 通常の $L^2$ 内積 となっている.

## 4. Eisenstein series

○ Siegel Eisenstein series $\in M_k(\Gamma_n)$

$k > n+1, \quad k$ : even

$$E_{n,k}(Z) := \sum_{\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in P_s \cap \Gamma_n \backslash \Gamma_n} \det(CZ+D)^{-k} \qquad P_s = \left\{ \begin{pmatrix} * & * \\ 0_n & * \end{pmatrix} \in G \right\}$$

Fourier exp. : Maass, Böecherer, Katsurada

○ Klingen Eisenstein series $\in M_k(\Gamma_n)$

$0 \le r < n$, $k > n+r+1$, $k$ : even

$\varphi \in S_k(\Gamma_r)$

$$E_{n,k}(\varphi : Z) := \sum_{M \in P_K \cap \Gamma_n \backslash \Gamma_n} \varphi(M\langle Z \rangle^*) \det(CZ+D)^{-k}$$

$M\langle Z\rangle^*$ : left $r \times r$ block of $M\langle Z \rangle$

$$P_K = \left\{ \begin{pmatrix} * & * \\ 0_{n-r,n+r} & * \end{pmatrix} \in G \right\}$$

Fourier expansion by Mizumoto

non holomorphic Eisenstein series

$$E_k(Z, s) = \sum_{(C,D)} \frac{\det \mathrm{Im}(Z)^s}{|\det(CZ+D)|^{2s}} \det(CZ+D)^{-k}$$

## 5. Hecke algebra & L-function

$G > \Gamma$, $G \supset \Delta$, $K \subset \mathbb{C}$ subring

multiplicatively closed.

$(\Gamma, \Delta)$ Hecke pair.

$H(\Gamma, \Delta)$ Hecke alg = free $K$-module generated by $\Gamma \alpha \Gamma$ $(\alpha \in \Delta)$

with product.

$$\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i \qquad \Gamma\beta\Gamma = \bigsqcup_j \Gamma\beta_j$$

$$(\Gamma\alpha\Gamma)\cdot(\Gamma\beta\Gamma) = \sum_\gamma C_\gamma \Gamma\gamma\Gamma$$

$$C_\gamma = \#\{(i,j) \mid \Gamma\alpha_i\beta_j = \Gamma\gamma\}$$

$$K = \mathbb{Q}$$

$$H = H(GSp_{2n}^+(\mathbb{Q}), \Gamma_n) \qquad \nu(g) > 0$$

$$H_p = H(GSp_{2n}^+(\mathbb{Q}) \cap GL_{2n}(\mathbb{Z}[\tfrac{1}{p}]), \Gamma_n)$$

$H$ is generated by $\{H_p \mid p : \text{prime}\}$

$$H_p = \mathbb{Q}[S, T_0^\pm, T_1, \cdots, T_{n-1}]$$

$$S = \Gamma_n \begin{bmatrix} \mathbb{1}_n & \\ & p\mathbb{1}_n \end{bmatrix} \Gamma_n$$

$$T_i = \Gamma_n \begin{bmatrix} \mathbb{1}_i & & & \\ & p\mathbb{1}_{n-i} & & \\ & & p^2\mathbb{1}_i & \\ & & & p\mathbb{1}_{n-i} \end{bmatrix} \Gamma_n \qquad 0 \le i \le n-1$$

$$T_0^+ = T_0, \quad T_0^- = \Gamma_n \begin{bmatrix} 1/p\,\mathbb{1}_n & \\ & 1/p\,\mathbb{1}_n \end{bmatrix} \Gamma_n$$

$$H_p \cong \mathbb{Q}[X_0^\pm, \cdots, X_n^\pm]^{W_n}$$

$\boxed{\text{Satake isom.}}$

per of $X_1 \sim X_n$

$$W_n = \left\langle \sigma, \tau_i \;\middle|\; \begin{matrix} \sigma \in \mathfrak{S}_n \\ 1 \le i \le n \end{matrix} \right\rangle$$

$$\tau_i(X_j) = \begin{cases} X_0 X_i & j = 0 \\ X_i^{-1} & j = i \\ X_j & j \ne 0, i \end{cases}$$

$$H_p \ni \Gamma_n g \Gamma_n = \bigsqcup_i \Gamma_n g \longrightarrow \sum_i X_0^{\delta_i} \left( \prod_{j=1}^{n} \frac{X_j}{p^j} \right)^{d_{ij}} \quad ⑦$$

$$g_i = \begin{pmatrix} p^{\delta_i}\, {}^tA_i^{-1} & B_i \\ * & A_i \end{pmatrix}, \quad A_i = \begin{pmatrix} p^{d_{i1}} & & * \\ & \ddots & \\ 0 & & p^{d_{in}} \end{pmatrix}$$

$n=1$

$$S = \Gamma_1 \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_1 = \bigsqcup_{\substack{ad=p \\ 0 \le b < d}} \Gamma_1 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \begin{array}{l} \tau(X_0) = X_0 X_1 \\ \tau(X_1) = X_1^{-1} \end{array}$$

$$= \bigsqcup_{0 \le b < p} \Gamma_1 \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \sqcup \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

$$X_0 \left( \frac{X_1}{p} \right) p \longmapsto X_0 X_1 + X_0$$

$$T_0^+ = \Gamma_1 \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \longrightarrow \frac{1}{p} X_0^2 X_1, \quad T_0^- \to p (X_0^2 X_1)^{-1}$$

$F \in M_\rho(\Gamma_n)$

$$T = \Gamma_n g \Gamma_n = \bigsqcup_i \Gamma_n g_i \in H_p$$

$$F|_T := \nu(g)^{m - \frac{n(n+1)}{2}} \sum_i f|_\rho g_i$$

$$(f|_\rho g)(Z) = \rho(J(g, Z)) f(g\langle Z \rangle)$$

$$g \in \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_{2n}^+(\mathbb{R})$$

$$J(g, Z) = (\det g)^{-\frac{1}{2}} (CZ + D)$$

$$\rho(\mathrm{diag}(t, \dots, t)) = t^{2m}\, id_V \qquad \rho = \det{}^k \rightsquigarrow m = \frac{nk}{2}$$

Assume $F$ is Hecke eigenform.

$$F|_T = \lambda_F(T)F \qquad \forall T \in H_p$$

$$\mathcal{S} : H_p \xrightarrow{\sim} \mathbb{Q}[X_0^{\pm}, \cdots, X_n^{\pm}]^{W_n}$$

$$\alpha_{i,p} = \lambda_F(\mathcal{S}^{-1}(X_i)) \cdots \text{ Satake } p\text{-parameter.}$$

$$0 \le i \le n.$$

<u>Def</u> $$L(s, F, \mathrm{spin}) := \prod_p \Big[ \prod_{r=0}^{n} \prod_{1 \le i_1 < \cdots < i_r \le n} (1 - \alpha_{0,p}\alpha_{i_1,p} \cdots \alpha_{i_r,p} p^{-s})^{-1} \Big]$$

$\uparrow$

$\deg = 2^n$

$$L(s, F, \mathrm{std}) := \prod_p \Big[ (1-p^{-s})^{-1} \prod_{1 \le i \le n} \{(1-\alpha_{i,p}p^{-s})(1-\alpha_{i,p}^{-1}p^{-s})\} \Big]^{-1}$$

$\uparrow$

$\deg = 2n+1$

$n=1$ $\quad L(s, F, \mathrm{spin}) = L(s, F)$ $\quad$ std $L$.

$$L(s, F, \mathrm{std}) = L(s, F, \mathrm{sym}^2)$$
$$= \zeta(2s) \sum_{n=1}^{\infty} \frac{a(n^2)}{n^{s+k-1}} \qquad F(z) = \sum a(n) e^{2\pi}$$

$$L(s, F, \mathrm{std}) = \boxed{\phantom{x}} \sum_{M \in SL_n(\mathbb{Z}) \backslash M_n^+(\mathbb{Z})} \frac{a_F(M^t M)}{(\det M)^{s+k-1}}$$

$$F \in S_k(\Gamma_n)$$

<u>Known results</u>

- analytic continuation of $L(s, F, \mathrm{std})$ $\quad \forall n$.

  $L(s, F, \mathrm{spin})$ $\quad n \le 3$ ?

$n=3$ Asgari-Schmidt

bad factor の定義

Std $L$ の bad factor ← Lapid - Rallis.

$$\frac{L(s, \pi, \mathrm{spin})}{L(s+1, \pi, \mathrm{spin})} \rightsquigarrow \text{const term of Klingen Eisenstein for } F_4$$

for $n=3$

↑

Asgari - Schmidt.