北山秀隆

Local newforms for $GSp(4)$

(2010/8/5)

Notation

F : a non-arch. local field of char $=0$

$\mathcal{O}$: the ring of integers of F

$\mathfrak{p} = \varpi \mathcal{O}$ the max. ideal of $\mathcal{O}$ $\qquad q = \#(\mathcal{O}/\mathfrak{p})$

$$GSp(4,F) := \left\{ g \in GL(4,F) \,\middle|\, {}^t g J g = \lambda(g) J,\ J = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \right\}$$

GL(2)-theory

$$\boxed{\Gamma_0(\mathfrak{p}^n)} = GL(2;F) \cap \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}^\times$$

$\downarrow$

??

$$K(\mathfrak{p}^n) := GSp(4,F) \cap \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-n} \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathcal{O} \end{pmatrix}^\times$$

$n \in \mathbb{Z}_{\geq 0}$

Ref [1] Roberts & Schmidt, " —— " (LNM 1918)

[2] Ibukiyama (Adv. Stud. in Pure Math. 1985)

§1. Main Theorem

(π, V) an irred. adm. rep of $GSp(4,F)$ with trivial central char.

$n \in \mathbb{Z}_{\geq 0}$

$$V(n) := V^{K(\mathfrak{p}^n)} = \{ v \in V \mid \pi(g)v = v \text{ for all } g \in K(\mathfrak{p}^n) \}$$

Rem $m \neq n \Rightarrow V(m) \not\subset V(n)$

1.1 newforms thm

Thm 1 We assume π is generic.

Then

(1) There exists $n \in \mathbb{Z}_{\geq 0}$ s.t $V(n) \neq 0$

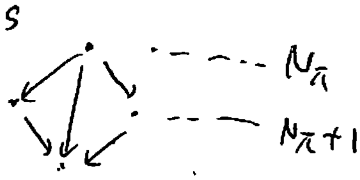
(2) If N_π is the smallest n s.t. $V(n) \neq 0$

then $\dim V(N_\pi) = 1$

Rem (1) does not hold for some non-generic rep's.

If (1) holds, then (2) also holds.

1.2. Old forms



- Atkin-Lehner involution

We put $u_n := \begin{pmatrix} & & 1 & \\ & & & -1 \\ \varpi^n & & & \\ & -\varpi^n & & \end{pmatrix}$ $n \in \mathbb{Z}_{\geq 0}$

Then $\pi(u_n) : V(n) \longrightarrow V(n)$

$(u_n K(\mathfrak{p}^n) = K(\mathfrak{p}^n) \cdot u_n)$

- level raising operator

$\theta' : V(n) \longrightarrow V(n+1)$

$$\theta' v := \frac{1}{\mathrm{vol}(K(\mathfrak{p}^n) \cap K(\mathfrak{p}^{n+1}))} \int_{K(\mathfrak{p}^{n+1})} \pi(k) v \, dk$$

$\vartheta : V(n) \longrightarrow V(n+1)$

$$\vartheta = \pi(u_{n+1}) \circ \vartheta' \circ \pi(u_n)$$

$\eta : V(n) \longrightarrow V(n+2)$

$$\eta \cdot v = \pi\left(\underbrace{\begin{pmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{pmatrix}}_{\gamma}\right) v \quad \left(\because K(\mathfrak{p}^{n+2})\gamma \subset \gamma K(\mathfrak{p}^n)\right)$$

(3) Let $W_\pi \in V(N_\pi)$.

If $n \geq N_\pi$, then $V(n)$ is spanned by the linearly indep. vector.

$$\vartheta'^{i} \vartheta^{j} \eta^{k} W_\pi \qquad \begin{array}{l} i, j, k \in \mathbb{Z}_{\geq 0} \\ i+j+2k = n - N_\pi \end{array}$$

$$\rightsquigarrow \dim V(n) = \left[\frac{(n - N_\pi + 2)^2}{4}\right]$$

Rem η does not have an analogue in GL(2)-theory

- η principle

For $W (\neq 0) \in V(n)$

$$Z(s, W) = 0 \iff \begin{cases} n' \geq 2 \\ \exists W' \in V(n-2) \text{ s.t. } W = \eta W' \end{cases}$$

Novodvorsky's Zeta integral

(Why η is necessary?)

By a relation between zeta integral and level raising operator,

$\forall W \in V(n)$

$\exists W' \in \{\theta^i \theta'^j W_\pi \mid i+j = n - N_\pi,\ i,j \in \mathbb{Z}_{\geq 0}\}$

s.t. $Z(s, W) = Z(s, W')$

By η-principle

$$W = W' + \eta W'', \quad \exists W'' \in V(n-2)$$

$\downarrow$ repeat.

To prove (1), (2)

(I) P_3 - theory

(II) zeta integral

(III) Hecke operator

(IV) Classification of irred. adm rep of $GSp(4, F)$

$\nearrow$ super cuspidal

$\searrow$ non-super cuspidal $\rightleftarrows$ 25 types

$\downarrow$ Sally & Tadić

§2. P_3 - theory

$$P_n = GL(n, F) \cap \begin{pmatrix} & * & \\ 0 & \cdots 0 & 1 \end{pmatrix}$$

(π, W) a rep of $GL(n, F)$

$\wr$ rep

$(\tau|_{P_n}, W)$ $\quad P_n$

$GSp(4)$ - case

Klingen

$$Q := GSp(4,F) \cap \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & u \end{pmatrix}$$

$$\square / u \in P_3$$

we have $Q / Z^J \cdot Z \xrightarrow{\sim} P_3$

where Z the center of $GSp(4)$

$$Z^J = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

(π, V) : an irred. adm rep of $GSp(4,F)$

$\rightsquigarrow$

$(\hat{\pi}, \hat{V})$ a smooth rep of P_3

(Bernstein-Zelevinsky)

where $\hat{V} = V / V(Z^J)$

$$V(Z^J) := \langle \{ v - \pi(g)v \mid v \in V,\ g \in Z^J \} \rangle$$

Key (1) $v \in V$ is a non-zero paramodular invariant vector

$\Rightarrow \hat{v}$ is a non-zero $P_3(\mathcal{O})$ invariant vector

(2) π is generic $\iff$ $c\text{-}\mathrm{Ind}_{U_3}^{P_3}(\psi) \subset \hat{V}$

where $U_3 = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$, $\Theta\left(\begin{pmatrix} 1 & u_{12} & * \\ & 1 & u_{23} \\ & & 1 \end{pmatrix}\right) := \psi(u_{12} + u_{23})$

Proof of "existence"

Let (π, V) generic

Step 1 ; We construct an "appropriate" $P_3(\theta)$

in $c\text{-}\mathrm{Ind}_{U_3}^{P_3}(\Theta)$

Step 2 We take a $v \in V$ s.t $v \mapsto \hat{v}$

and modify it by

$\int_{Q(\theta)} \pi(k) v \, dk$ and $\theta' : V \to V(n)$

Then it is non-zero paramodular.

invariant.

§3. Uniqueness (generic supercuspidal)

(π, V)

In this case,

- π is "paramodular" ($\Leftrightarrow$ (1) holds)
- $L(s, \pi) = 1$ $\left(\Leftrightarrow Z(s, W) \in \mathbb{C}[q^{-s}, q^{s}] \quad {}^{\forall} W \in V\right)$

N_π : the minimal level s.t $V(N_\pi) \neq 0$

Hecke operator on $V(N_\pi)$

$$T_{h_1} \qquad K(\mathfrak{p}^{N_\pi})\begin{pmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} K(\mathfrak{p}^{N_\pi})$$

$$T_{h_2} \qquad K(\mathfrak{p}^{N_\pi})\begin{pmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{pmatrix} K(\mathfrak{p}^{N_\pi})$$

By direct calculation,

$$\begin{cases} T_{h_1} T_{h_2} = T_{h_2} T_{h_1} \\ T_{h_1}, T_{h_2} \text{ are diagonalizable} \end{cases}$$

<u>Fact</u> ([RS] Prop. 7.4.5)

We put $T_{h_1} W = \lambda W$, $T_{h_2} W = \mu W$

Then. $$Z(s, W) = \frac{(1-q^{-1}) \cdot W(1)}{1 - q^{-3/2}\lambda \cdot q^{-s} + (q^{-2}\mu + 1) q^{-2s}}$$

Since $L(s,\pi) = 1$, we have $\lambda = 0$, $\mu = q^2$

Hence $Z(s, W) = (1-q^{-1}) W(1) \in \mathbb{C}$ for $\forall W \in V(N_\pi)$

For $W, W' \in V(N_\pi)$

$$W(1) = W'(1) \Rightarrow Z(s, W - W') = 0$$
$$\Rightarrow W = W'$$

§4. Uniqueness (generic non-supercuspidal)

$$B = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}, \quad P = \begin{pmatrix} \square & * \\ 0 & \square \end{pmatrix}, \quad Q = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

non-supercuspidal 25 types

$\left\{ \begin{array}{l} \cdot \text{ Induction from } B, P \text{ or } Q \text{ which is irred.} \\ \qquad \text{or} \\ \cdot \ 0 \longrightarrow \oslash \longrightarrow \mathrm{Ind} \longrightarrow \square \longrightarrow 0 \end{array} \right.$

To calculate $\dim V^{K(\mathfrak{p}^n)}$ for all (π, V)

<u>Step 1</u> We calculate all $\dim \left(\mathrm{Ind}_K^G(\cdot)\right)$

(by double coset decomp, $GL(2)$-theory)

<u>Step 2</u> $0 \longrightarrow \oslash^{K(\mathfrak{p}^n)} \longrightarrow (\mathrm{Ind})^{K(\mathfrak{p}^n)} \longrightarrow \square^{K(\mathfrak{p}^n)} \longrightarrow 0$

If we know $\oslash^{K(\mathfrak{p}^n)}$ or $\square^{K(\mathfrak{p}^n)}$, then

we obtain the other by subtraction

$\left(\begin{array}{l} \cdot \text{ non paramodular} \\ \cdot \text{ generic} \ \longleftarrow \ \text{Thm } 1(3) \quad \text{Old forms} \end{array} \right)$