p-adic L-functions for Galois deformation spaces and Iwasawa Main Conjecture

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Main Reference

[1] "A generalization of the Coleman map for Hida deformation", the American Journal of Mathematics, 2003.
[2] "Euler system for Galois deformation", Annales de l'institut Fourier, 2005.
[3] "On the two-variable Iwasawa Main Conjecture for Hida deformations", preprint 2004.

Contents of the talk

★ General problems on p-adic L-functions
★ Two-variable p-adic L-functions for Hida families

<u>Situation</u>

p: fixed odd prime number,

$\mathbb{Q}_\infty$: the cyclotomic $\mathbb{Z}_p$-ext. of $\mathbb{Q}$

$$\Gamma := \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \xrightarrow[\chi]{\sim} 1 + p\mathbb{Z}_p \subset (\mathbb{Z}_p)^\times$$

(χ: p-adic cyclo. char)

We recall examples of p-adic L-functions.

Example 1.

Theorem(K-L, I, C).

ψ: Dirichlet Character of Conductor $D > 0$ with $(D, p) = 1$

$\exists L_p(\psi) \in \mathbb{Z}_p[\psi][[\Gamma]]$ such that

$$\chi^r(L_p(\psi)) = (1 - \psi(p)p^r)L(\psi, -r)$$

for each $r \geq 0$ divisible by $p-1$.

Example 2.

E: an ellip. curve defined over $\mathbb{Q}$.

$L(E,s)$: Hasse-Weil L-function for E.

Theorem(M-S).

If E has good ordinary red. at p, Then, $\exists L_p(E) \in \mathbb{Z}_p[[\Gamma]]$ such that

$$\phi(L_p(E)) = \left(1 - \frac{\phi(p)}{\alpha}\right)^2 \times$$

$$\alpha^{-s(\phi)} G(\phi^{-1}) \frac{L(E,\phi,1)}{\Omega_E^+}$$

for every finite order char. ϕ on Γ where $s(\phi) = \mathrm{ord}_p \mathrm{Cond}(\phi)$, α: p-unit root of $x^2 - a_p(E)x + p = 0$ with $a_p(E) = 1 + p - \sharp E_p(\mathbb{F}_p)$

$G(\phi^{-1})$:Gauss sum,$\Omega_E^+ = \int_{E(\mathbb{R})} \omega_E$,

Translation $L_p(E)$ is defined on
$\mathcal{X} := \{\text{cont. char's } \Gamma \xrightarrow{\eta} \overline{\mathbb{Q}}_p^\times\}$
$\cong$ a unit ball $U(1;1) \subset \overline{\mathbb{Q}}_p$
$(U(a;r) = \{x \in \overline{\mathbb{Q}}_p ; |x-a|_p < r\})$

$\widetilde{T} := T_p(E) \otimes \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ where
$\widetilde{\chi} : G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \mathbb{Z}_p[[\Gamma]]^\times$
$\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$: free $\mathbb{Z}_p[[\Gamma]]$-module of rank one on which $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via $\widetilde{\chi}$

Then the specialization $\widetilde{T}_\phi := \widetilde{T} \otimes_{\mathbb{Z}_p[[\Gamma]]} \mathbb{Z}_p[\phi]$ at $\phi \in \mathcal{X}$ is isomorphic to $T_p(E) \otimes \phi$. $L_p(E)$ is associated to $\widetilde{T}$.

Consider the following situations:

$\bigstar\mathcal{B}$: a rigid anal. space over $\mathbb{Q}_p$

(Mostly, we think of a finite cover of an open unit ball in $\overline{\mathbb{Q}}_p^{\oplus s}$)

$\mathcal{B} = \mathrm{Spf}\mathcal{O}(\mathcal{B})$

(If $\mathcal{B} = \mathcal{X}$, $\mathcal{O}(\mathcal{X}) = \mathbb{Z}_p[[\Gamma]]$)

$\bigstar\mathcal{T}$: a family of Galois representations over $\mathcal{B}$

(Mostly, $\mathcal{T} \cong \mathcal{O}(\mathcal{B})^{\oplus d}$)

$\bigstar P$: a dense subset in $\mathcal{B}$ such that $\mathcal{T}_x \cong H_{\text{ét},p}(M_x) \curvearrowright G_{\mathbb{Q}}$ at each $x \in P$ for a certain motive M_x which is critical in the sense of Deligne.

Recall that

A (pure) motive M over $\mathbb{Q}$ called <u>critical</u> if the composite
$H_{\mathsf{B}}^{+}(M) \otimes \mathbb{C} \hookrightarrow H_{\mathsf{B}}(M) \otimes \mathbb{C} \xrightarrow{\sim} H_{\mathsf{dR}}(M) \otimes \mathbb{C} \twoheadrightarrow \mathrm{Fil}^0 H_{\mathsf{dR}}(M) \otimes \mathbb{C}$
is an isomorphism, where $H_{\mathsf{B}}^{+}(M)$ is $+$-part for the action of the complex conj. on the Betti realization $H_{\mathsf{B}}(M)$.

Deligne's conjecture.
$L(M, 0)/\Omega_M^{+} \in \overline{\mathbb{Q}}$,
(where $\Omega_M^{+} \in \mathbb{C}$ is the det. of
$H_{\mathsf{B}}^{+}(M) \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{Fil}^0 H_{\mathsf{dR}}(M) \otimes \mathbb{C}$).

Examples.

$\bigstar M = \mathbb{Q}(r)$: Tate Motive

$L(M, s) = \zeta(s + r)$ is the Riemann's zeta function.

M is critical $\Leftrightarrow r = 2n$ or $1-2m$ with $n, m \in \mathbb{Z}_{>0}$.

$\bigstar M = M_f(j)$: j-th Tate twist of the motive for an eigen cuspform f of weight $k \geq 2$

$L(M_f(j), s) = L(f, s + j)$ Hecke L-funct. for f

$M_f(j)$ is critical $\Leftrightarrow 1 \leq j \leq k-1$

We call $(\mathcal{B}, \mathcal{T}, P)$ a geometric triple. For a given $(\mathcal{B}, \mathcal{T}, P)$, consider:

Problem. Is there a function $L_p(\mathcal{T})$ on $\mathcal{B}$ with p-adic continuity which is characterized by the following interpolation property: $L_p(\mathcal{T})(x) = N_x \times L(M_x, 0)/\Omega^{+}_{M_x}$ at each $x \in P$?

(N_x is a certain "normalization factor" at x)

Remarks.

•normalization factors are related to p-adic periods at x, Euler like factor, Gauss sum etc.

•We have to specify the algebra $R \supset \mathcal{O}(\mathcal{B})$ where $L_p(\mathcal{T})$ is contained.

Example.

$\mathcal{B} = \mathcal{X}$, $\mathcal{T} = T_p(E) \otimes \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$

E: supersingular at p. We have $L_p(E)$ with the same interpolation property as ordinary cases. $L_p(E)$ is never contained in $\mathcal{O}(\mathcal{X})$, but is contained in a larger ring $\mathcal{H}_1 \supset \mathcal{O}(\mathcal{X})$.

To give a result convincing to formulate the general conjecture, the following Hida deformations are important.

Preparation.

Γ': the group of Diamond operators on the tower $\{Y_1(p^t)\}_{t\geq 1}$ of modular curves

$$\Gamma' \xrightarrow[\chi']{\sim} 1 + p\mathbb{Z}_p \subset (\mathbb{Z}_p)^\times$$

$$\mathcal{Y} := \{\text{cont. char's } \Gamma' \xrightarrow{\eta'} \overline{\mathbb{Q}}_p^\times\}$$

$$\cong \text{a unit ball } U(1;1) \subset \overline{\mathbb{Q}}_p$$

Hida families.

★a finite cover $\mathcal{B} \xrightarrow{q} \mathcal{X} \times \mathcal{Y}$.

★$\mathcal{T}$ is a family of Galois representations on $\mathcal{B}$ which is generically of rank two. (Mostly, $\mathcal{T} \cong \mathcal{O}(\mathcal{B})^{\oplus 2}$)

★P consists of $x \in \mathcal{B}$ such that $q(x)|_U = \chi^{j(x)} \times {\chi'}^{k(x)}$ satisfying
$1 \leq j(x) \leq k(x) - 1$
for a certain open subgroup $U \subset \Gamma \times \Gamma'$.

$(\mathcal{B}, \mathcal{T}, P)$ is a geometric triple with the following properties:

For each $x \in P$,

$\bullet \exists f_x$: an ordinary eigen cusp-form of weight $k(x)$

$\bullet \exists \phi_x$ a finite order character of Γ s. t. $\mathcal{T}_x \cong T_p(f_x)(j(x)) \otimes \phi_x \omega^{-j(x)}$.

($T_p(f)$: rep. of $G_{\mathbb{Q}}$ asso. to f, ω: the Teichmuller character)

Known constructions of (candidates of) p-adic L-functions for $(\mathcal{B}, \mathcal{T}, P)$ are classified into three cases below:

- Use the theory of complex multiplication. (Only for $\mathcal{T}$ with CM/by Katz, Yager, etc)
- Use the theory of modular symbols (Kitagawa, Greenberg-Stevens, etc)
- Use the Eisenstein family and Shimura's theory (Panchishkin, Fukaya, Ochiai, etc)

$L_p^{\mathsf{Ki}}(\mathcal{T}) \in \mathcal{O}(\mathcal{B})$ is rather desirable so that

$\exists U$ an invertible element in $(\mathcal{O}(\mathcal{B}) \hat{\otimes} \mathcal{O}_{\mathbb{C}_p}) \supset \mathcal{O}(\mathcal{B})$ such that $L_p^{\mathsf{Ki}}(\mathcal{T}) \cdot U \in \mathcal{O}(\mathcal{B}) \hat{\otimes} \mathcal{O}_{\mathbb{C}_p}$ has the interpolation property:

$$(L_p^{\mathsf{Ki}}(\mathcal{T})(x) \cdot U(x))/\Omega_{p,x}^{+} =$$

$$(-1)^{j-1}(j-1)! \left(1 - \frac{\phi_x \omega^{-j}(p) p^{j-1}}{a_p(f_x)}\right) \times \left(\frac{p^{j-1}}{a_p(f_x)}\right)^{s(j)} \frac{L(f_x, \phi_x \omega^{-j}, j)}{\Omega_{\infty,x}^{+}}$$

at each $x \in P$.

Remark.

•$\Omega^{+}_{p,x} \in \mathbb{C}_p$ is the p-adic period at x defined to be the determinant of:

$$H_{\mathsf{B}}(M_{f_x})^{+} \otimes B_{\mathsf{HT}} \xrightarrow{\sim} \mathsf{Fil}^0 H_{\mathsf{dR}}(M_{f_x}) \otimes B_{\mathsf{HT}}.$$

•This interpolation uniquely characterizes the ideal $(L_p^{\mathsf{Ki}}(\mathcal{T})) \in \mathcal{O}(\mathcal{B})$

•By using both of $\Omega^{+}_{p,x}$ and $\Omega^{+}_{\infty,x}$, the interpolation is <u>balanced</u> so that it is independent of the choice of bases.

From our detailed study on Selmer groups for $\mathcal{T}$, we have a well-chosen the algebraic p-adic L-function $L_p^{\mathrm{alg}}(\mathcal{T})$ defined to be the characteristic ideal of certain Selmer group for $\mathcal{T}$. Thus, we propose

Iwasawa Main Conjecture.
$(L_p^{\mathrm{Ki}}(\mathcal{T})) = (L_p^{\mathrm{alg}}(\mathcal{T}))$ (refinement of the conj. by Greenberg)

Theorem(O-).

We have the interpolation map:

$\Xi : H^1_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1)) \longrightarrow \mathcal{O}(\mathcal{B})$ with

$\exp_x^* \circ x \fallingdotseq x \circ \Xi$

where

$\bullet \mathcal{T}^*(1) = \mathrm{Hom}_{\mathcal{O}(\mathcal{B})}(\mathcal{T}, \mathcal{O}(\mathcal{B})(1))$

$\bullet \exp_x^*$ is the dual exponential map $H^1_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1)) \longrightarrow \overline{\mathbb{Q}}_p$.

$\mathcal{Z} \in H^1(\mathbb{Q}_p, \mathcal{T}^*(1))$: Kato's Euler system element such that

$$\exp^* \circ x = \frac{L_{(p)}(f_x, \phi_x, j(x))}{(2\pi\sqrt{-1})^{j-1} L(f_x, 1)}$$

$\Xi(\mathcal{Z}) \in \mathcal{O}(\mathcal{B})$ satisfies

$$(\Xi(\mathcal{Z}))(x) =$$

$$(-1)^{j-1}(j-1)! \left(1 - \frac{\phi_x \omega^{-j}(p) p^{j-1}}{a_p(f_x)}\right)$$

$$\times \left(\frac{p^{j-1}}{a_p(f_x)}\right)^{s(j)} \frac{L(f_x, \phi_x \omega^{-j}, j)}{(2\pi\sqrt{-1})^{j-1} L(f_x, 1)}$$

$\Xi(\mathcal{Z}) \in \mathcal{O}(\mathcal{B})$ gives the interpolation of the L-values, but the complex period is not "optimal". Later we arrived the following modification:

Theorem (O-). We have a normalized $\mathcal{Z}^{\mathsf{Ki}} \in H^1_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1))$ such that $\Xi(\mathcal{Z}^{\mathsf{Ki}}) = L_p^{\mathsf{Ki}}(\mathcal{T})$.

This theorem combined with Euler system theory for Galois deformations gives:

Theorem (O-).
$(L_p^{\mathsf{Ki}}(\mathcal{T})) \subset (L_p^{\mathsf{alg}}(\mathcal{T}))$