宮内通孝

保型表現のdepth概説及び
$GSp(4)$の場合のdepth

§1. Depth of rep's of $GL(N)$
§2. Depth and normalized level
§3. Depth and conductor
§4. Depth of rep's of $Sp(4)$

$$G = GL_N(F) \curvearrowright \mathcal{I} : \text{building of } G$$
$$\mathcal{I} \ni x$$

$G_x = \mathrm{Stab}_G\, x$: parabolic subgp

$$\text{Moy-Prasad} \longrightarrow \{G_{x,r}\}_{r \in \mathbb{R}_{\geq 0}}$$

$$G_{x,r+} = \lim_{s \to r+} G_{x,s}$$

π : irred smooth rep of G

$$\mathrm{depth}(\pi) = \inf\{ r \mid \pi^{G_{x,r+}} \neq \{0\}, \text{ for some } x \}$$

π : irred supercusp. rep of G

conductor of π = $N \cdot \mathrm{depth}\,\pi + N \geq N$

F : non-arch. local field.
$\cup$
$\mathcal{O}$: ring of integers
$\cup$
$\mathfrak{p} = \varpi \mathcal{O}$ max. ideal v_F : valuation of F s.t. $v_F(\varpi) = 1$

$k_F = \mathcal{O}/\mathfrak{p}$, $q = \# k_F$

$V = F^N$, $A = \mathrm{End}_F(V) \simeq M_N(F)$

$G = A^\times \simeq GL_N(F)$, $\mathrm{Lie}\, G \simeq A$

§1. Depth of GL(N)

<u>Def</u> A norm on V is a map

$\alpha : V \longrightarrow \mathbb{R} \cup \{\infty\}$ s.t.

(i) $\alpha(xv) = v_F(x) + \alpha(v)$ $\quad x \in F,\ v \in V$

(ii) $\alpha(v+w) \geq \inf\{\alpha(v), \alpha(w)\}$ $\quad v, w \in V$

(iii) $\alpha(x) = \infty \iff x = 0$

$\mathrm{Norm}'(V)$: the set of norms on V

$G \curvearrowright \mathrm{Norm}'(V)$

$(g\alpha)(v) = \alpha(g^{-1}v)$ $\quad g \in G$

$\alpha \in \mathrm{Norm}'(V)$, $v \in V$

$I' = I'(G, F)$: building of G.

There exists a G-set bijection.

$I' \simeq \mathrm{Norm}'(V)$ (Bruhat-Tits)

<u>Def'n</u> A lattice function in V is a map

$\Lambda : \mathbb{R} \longrightarrow \{\mathcal{O}\text{-lattices in } V\}$ s.t

(rank N $\mathcal{O}$-modules)

(i) $\varpi \Lambda(r) = \Lambda(r+1)$, $r \in \mathbb{R}$

(ii) $\Lambda(r) \supset \Lambda(s)$ if $s > r$.

(iii) Λ is left continuous

$\forall r \in \mathbb{R}$ $\exists \varepsilon > 0$ s.t. Λ is const. on $(r - \varepsilon, r]$

There are only finite $r \in \mathbb{R}/\mathbb{Z}$ s.t. $\Lambda(r) \neq \Lambda(r+)$ $\quad \left(\Lambda(r+) = \bigcup_{s>r} \Lambda(s) \right)$

$\mathrm{Latt}^1(V)$ the set of lattice functions

$G \curvearrowright \mathrm{Latt}^1(V)$

$$(g\Lambda)(r) = g\Lambda(r) \qquad g \in G$$

$$\Lambda \in \mathrm{Latt}^1(V), \quad r \in \mathbb{R}$$

$$\mathrm{Norm}^1(V) \longrightarrow \mathrm{Latt}^1(V) \qquad \alpha \longmapsto \Lambda_\alpha$$

$$\Lambda_\alpha(r) = \{ v \in V \mid \alpha(v) \geq r \}$$

$$\mathrm{Latt}^1(V) \longrightarrow \mathrm{Norm}^1(V) : \Lambda \longrightarrow \alpha_\Lambda$$

$$\alpha_\Lambda(v) = \sup \{ r \mid v \in \Lambda(r) \}$$

Prop (Broussous - Lemaire)

The map $\mathrm{Norm}^1(V) \to \mathrm{Latt}^1(V)$; $\alpha \mapsto \Lambda_\alpha$ is a G-set bijection.

$$I^1 \simeq \mathrm{Norm}^1(V) \simeq \mathrm{Latt}^1(V)$$

essentially in 「Basic Number Theory」 A. Weil

For $\Lambda \in \mathrm{Latt}^1(V)$,

$$\mathfrak{a}_{\Lambda, r} = \{x \in A \mid x\Lambda(s) \subset \Lambda(s+r),\ s \in \mathbb{R}\}$$

The map $r \mapsto \mathfrak{a}_{\Lambda, r}$ is a lattice function in A

$$\mathfrak{a}_{\Lambda, r+} = \bigcup_{s>r} \mathfrak{a}_{\Lambda, s}$$

Put $U_\Lambda = U_{\Lambda, 0} = \{g \in G \mid g\Lambda = \Lambda\}$
parahoric subgp

$$U_{\Lambda, r} = 1 + \mathfrak{a}_{\Lambda, r}, \quad U_{\Lambda, r+}$$

$U_{\Lambda, r}$ is an open normal subgp of U_Λ

<u>Def</u> π : irred. smooth rep. of G

$$\mathrm{depth}(\pi) = \inf\{r \mid \pi^{U_{\Lambda, r+}} \neq \{0\} \text{ for some } \Lambda\}$$

$$\geqq 0$$

<u>Prop</u> (Moy - Prasad)

$$\mathrm{depth}(\pi) \in \mathbb{Q}_{\geq 0}$$

§2. Depth and normalized level

A lattice chain in V is a set $\mathcal{L} = \{L_i \mid i \in \mathbb{Z}\}$ of $\mathcal{O}$-lattices in V s.t.

(i) $L_i \supsetneqq L_{i+1} \quad i \in \mathbb{Z}$

(ii) $\exists\, e \in \mathbb{N}$ s.t. $\varpi L_i = L_{i+e}, \ i \in \mathbb{Z}$

$$\mathfrak{A} = \{x \in A \mid xL_i \subset L_i \quad i \in \mathbb{Z}\}$$

hereditary order ass. to $\mathcal{L}$.

$$\mathfrak{P} = \{x \in A \mid xL_i \subset L_{i+1}, \ i \in \mathbb{Z}\}$$

Jacobson radical of $\mathfrak{A}$

filtration of A $\{\mathfrak{P}^k\}_{k \in \mathbb{Z}}$ of G.

$$U_{\mathfrak{A}} = U_{\mathfrak{A},0} = \mathfrak{A}^\times \qquad U_{\mathfrak{A},k} = 1 + \mathfrak{P}^k, \ k \geq 1$$

<u>Def</u> π: irred. smooth rep of G

$$\mathrm{level}(\pi) = \min\{n/e \mid \pi^{U_{\mathfrak{A},n+1}} \neq \{0\} \text{ for some } \mathfrak{A}\}$$

normalized level. $\qquad e \leq N$

<u>Fact</u> $\mathrm{depth}(\pi) = \mathrm{level}(\pi) \in \mathbb{Q}_{\geq 0}$

$\mathcal{L} = \{L_i \mid i \in \mathbb{Z}\}$ lattice chain

Put $\Lambda(r) = L_{\lceil r/e \rceil}$, $r \in \mathbb{R}$

$$\mathfrak{A}_{\Lambda,r} \neq \mathfrak{A}_{\Lambda,r+} \iff r \in \tfrac{1}{e}\mathbb{Z}$$

$\mathfrak{A}_{\Lambda,k/e} = \mathfrak{P}^k$, $k \in \mathbb{Z}$

$\mathfrak{A}_{\Lambda,(k/e)+} = \mathfrak{P}^{k+1}$

$\mathfrak{A}$: hereditary order corresp. to $\mathcal{L}$.

$\mathfrak{A}$ is called principal if $\mathfrak{P}$ is a principal ideal in $\mathfrak{A}$

$\Leftrightarrow \dim_{k_F} L_i/L_{i+1} = N/e$

<u>Prop</u> (Bushnell)

π : irred. super cusp. rep. of G

$\exists\, \mathfrak{A}$: principal order $\exists n$ s.t.

$\text{level}(\pi) = n/e$ and $\pi^{U_{\mathfrak{A}, n+1}} \neq \{0\}$

e : period of $\mathfrak{A}$.

<u>Rem</u> π : irred. super cusp. rep of G

$N \cdot \text{depth}(\pi) = N\, n/e \in \mathbb{Z}$

conductor of π = $N \cdot \text{depth}(\pi) + N$

§3. Depth and conductor

We recall the Godement-Jacquet funct. eq

$\psi : F \to \mathbb{C}$: char. $\psi|_{\mathfrak{o}} \equiv 1$

$\psi|_{\mathfrak{p}^{-1}} \not\equiv 1$

$\psi_A : A \to \mathbb{C} : x \mapsto \psi(\mathrm{tr}_{A/F}(x))$

$|x| = q^{-v_F(\det(x))}$ $\quad x \in A$

$\mathcal{S}(A) = \{ f : A \to \mathbb{C}$. loc. const. cpt supported $\}$

$\hat{\Phi}(y) = \int_A \Phi(x) \psi_A(xy)\, dx$, $\quad \Phi \in \mathcal{S}(A)$

$y \in A$

We can (and do) take Haar measure on A

$$\hat{\hat{\Phi}}(x) = \Phi(-x), \quad x \in A$$

(π, V) : irred super cusp. rep of G.

$(\hat{\pi}, \hat{V})$ contragradient of π

$$\langle \ , \ \rangle : \hat{V} \times V \longrightarrow \mathbb{C}$$

For $\Phi \in \mathcal{S}(A)$ and $s \in \mathbb{C}$

$$Z(\Phi, \pi, s) = \int_G \Phi(x)\pi(x)|x|^s d^\times x$$

$$\in \mathrm{End}_{\mathbb{C}}(V)$$

(converges for $\mathrm{Re}(s) \gg 0$)

$$\Xi(\Phi, \pi, s) = Z(\Phi, \pi, s + \tfrac{1}{2}(N-1))$$

functional equation

$${}^t\Xi(\hat{\Phi}; \hat{\pi}, 1-s) = \varepsilon(\pi, s)\,\Xi(\Phi, \pi, s)$$

transpose w.r.t. $\langle \ , \ \rangle$

$$\varepsilon(\pi, s) \in \mathbb{C}[q^s, q^{-s}]^\times$$

$$\varepsilon(\pi, s) = \varepsilon(\pi, 0)\, q^{-c(\pi)s} \qquad c(\pi) \in \mathbb{Z}$$

Thm (Bushnell - Fröhlich)

$$c(\pi) = N \underset{\substack{\shortparallel \\ \mathrm{depth}(\pi)}}{\mathrm{level}(\pi)} + N \geq N$$

§4. Depth of rep. of $Sp(4)$

Assume $\mathrm{char}(k_F) \neq 2$.

From now on, $V = F^4$,

h: alternating form on V

$$G = Sp_4(F) = \{ g \in GL_4(F) \mid h(gv, gw) = h(v,w),\ v, w \in V \}$$

For an $\mathcal{O}$-lattice L in V

$$L^{\#} = \{ v \in V \mid h(v, L) \subset \mathfrak{p} \}$$

<u>Def</u> A lattice funct. Λ in V is called self-dual if

$$\Lambda(r) = \Lambda((-r)+)^{\#}, \quad r \in \mathbb{R}$$

$\mathrm{Latt}_h^1(V)$, the set of self-dual lattice function.

$G \curvearrowright \mathrm{Latt}_h^1(V)$

I: building of G.

<u>Prop</u> (Broussous - Stevens)

There is a G-set bijection $I \simeq \mathrm{Latt}_h^1(V)$.

For $\Lambda \in \mathrm{Latt}_h^1(V)$

$$U_{\Lambda,0} = G \cap \mathfrak{a}_{\Lambda,0}$$

$$U_{\Lambda,r} = G \cap (1 + \mathfrak{a}_{\Lambda,r})$$

This filtration coincides with Moy-Prasad filtration (Lemaire, preprint)

π: irred smooth rep of G

$$\mathrm{depth}(\pi) = \inf\{r \mid \pi^{U_{\Lambda,r+}} \neq \{0\} \text{ for some } \Lambda \in \mathrm{Latt}_h^1(V)\} \in \mathbb{Q}_{\geq 0}$$

__Rem__ $G = Sp_4(F)$

$$4\,\mathrm{depth}(\pi) \in \mathbb{Z}$$