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代数群の代数的表現入門と $GSp(4)$ の表現の解説

§ Intro.

$\underline{G}$: reductive linear alg gp $/\mathbb{C}$

We consider fin. dim rep. of $\underline{G}$

fin. dim. hol rep of $\underline{G}(\mathbb{C})$
complex Lie gp

Motivation
- rep. of L-group
- Construction of loc constant sheaf

(rep. of max cpt subgp of $\underline{G}(\mathbb{C})$ or $\underline{G}(\mathbb{R})$)

§2. Lie gp and Lie alg.

G : semi-simple linear Lie gp

$\overset{def}{\iff}$ G is closed subgp of $GL(n,\mathbb{C})$

s.t $\begin{cases} \cdot\ g \in G \mapsto {}^t\bar{g} \in G \\ \cdot\ \text{center of } G \text{ is finite} \end{cases}$

$$\mathfrak{g} = \{X \in M_n(\mathbb{C}) \mid \exp(tX) \in G \quad (\forall t \in \mathbb{R})\}$$

Lie alg of G $\left(\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n\right)$

$\mathbb{K}$-vect. space with the bracket product.

$[X,Y] = XY - YX$ for $X, Y \in \mathfrak{g}$

$(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$

When $\mathbb{K} = \begin{cases} \mathbb{C} \\ \mathbb{R} \end{cases}$ we call G $\begin{cases} \text{complex} \\ \text{real} \end{cases}$ semi-simple lin Lie gp

$\rightsquigarrow$ G is complex manifold.

Ex ① $G = SL(n, \mathbb{C})$

$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) = \{ X \in M_n(\mathbb{C}) \mid tr(X) = 0 \}$

② $G = Sp(2n, \mathbb{C}) = \{ g \in GL(2n, \mathbb{C}) \mid {}^t g J_n g = J_n \}$,

$$J_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$$

$\mathfrak{g} = \mathfrak{sp}(2n; \mathbb{C}) = \{ X \in M_{2n}(\mathbb{C}) \mid {}^t X J_n + J_n X = 0 \}$

V: fin. dim vect space $/\mathbb{C}$

(Π, V) rep. of G on $V \rightsquigarrow (d\Pi, V)$ rep. of $\mathfrak{g}$ on V

$\mathbb{R}$-linear map $d\Pi : \mathfrak{g} \to End_{\mathbb{C}}(V)$

s.t. $d\Pi([X, Y]) = d\Pi(X) d\Pi(Y) - d\Pi(Y) d\Pi(X)$

for $X, Y \in \mathfrak{g}$

by $d\Pi(X) v = \left.\frac{d}{dt}\right|_{t=0} \Pi(e^{tX}) v \quad (X \in \mathfrak{g}, v \in V)$

§3. Weyl's unitary trick

Hereafter, let G be a complex semi-simple lin. Lie gp which is connected and simply connected

Ex $SL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$

Thm (Weyl's unitary trick)

G': real Lie subgp of G whose Lie alg $\mathfrak{g}'$ satisfies

$\mathfrak{g} = \mathfrak{g}' \oplus \sqrt{-1}\, \mathfrak{g}'$

Then we have the eq of the following categories:

(a) finite dim. hol. rep. of G

(b) finite dim $\mathbb{C}$-linear rep of $\mathfrak{g}$

(c) finite dim rep of G'

(d) finite dim rep of $\mathfrak{g}'$

Via
$$\begin{tikzcd} (a) \arrow[r, "\text{restriction}"'] & (c) \arrow[r, "\text{differential}"'] & (d) \arrow[r, "\mathbb{C}\text{-lin ext.}"'] & (b) \arrow[lll, bend left, "\text{We need the assumption } (*)"] \end{tikzcd}$$

Here G' is called real form of G.

<u>Example 1</u> $G' = G \cap U(n)$: cpt real form of $G \subset GL(n, \mathbb{C})$

(<u>Fact</u> Finite dim rep's of a cpt gp are completely reducible.
$\rightsquigarrow$ finite dim rep of G is completely reducible)

<u>Example 2</u> When $G = SL(n, \mathbb{C})$, $G' = SU(n)$, $SL(n, \mathbb{R})$

<u>Example 3</u> When $G = Sp(2n, \mathbb{C})$, $G' = Sp(2n)$, $Sp(2n, \mathbb{R})$

§4. Reps of $SL(2, \mathbb{C})$

$V_\ell = \mathrm{Sym}^\ell(\mathbb{C}^{\oplus 2})$: ℓ-th sym. power of $\mathbb{C}^2$

$\cong$ the space of deg ℓ homogeneous polynomials in $\mathbb{C}[e_1, e_2]$ (e_1, e_2 : standard basis of $\mathbb{C}^2$)

(Φ_ℓ, V_ℓ) hol rep of $SL(2, \mathbb{C})$ defined by

$$\Phi_\ell(g) f(e_1, e_2) = f(g \times e_1, g \times e_2)$$

$(g \in SL(2, \mathbb{C}),\ f(e_1, e_2) \in V_\ell)$

$$\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C}H_0 + \mathbb{C}E_+ + \mathbb{C}E_-$$

$H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Prop

① Φ_ℓ is irreducible

② (π, V) . irred. $(\ell+1)$-dim $\mathbb{C}$-linear rep of $\mathfrak{sl}(2,\mathbb{C})$

$\Longrightarrow \pi \simeq d\Phi_\ell$

(Pf) Key point

$\pi(X)\pi(Y) = \pi(Y)\pi(X) + \pi([X,Y])$ for $X, Y \in \mathfrak{sl}(2,\mathbb{C})$

Especially, $\pi(H_0)\pi(E_\pm) = \pi(E_\pm)(\pi(H_0) \pm 2)$

$\longrightarrow \pi(H_0)v = \lambda v \quad (\lambda \in \mathbb{C},\ v \in V)$

$\Longrightarrow \pi(H_0)\pi(E_\pm)v = (\lambda \pm 2)\pi(E_\pm)v$

Step 1

Find $0 \neq v_0 \in V$ s.t.

$\pi(H_0)v = \lambda v \ ({}^\exists\lambda \in \mathbb{C}),\ \pi(E_+)v_0 = 0$

Step 2

Evaluating the action of $\mathfrak{sl}(2,\mathbb{C})$, we have

- $V = \bigoplus_{i=0}^{\ell} \mathbb{C} v_i \quad (v_i = \pi(E_-)^i v_0)$
- $\lambda = \ell$
- $V \simeq V_\ell$ via $v_i \longleftrightarrow \dfrac{\ell!}{(\ell-i)!} e_1^{\ell-i} e_2^{i}$

§5. str. of $\mathfrak{g}$

$\mathfrak{h}$: Cartan subalg of $\mathfrak{g}$

$\underset{\text{def}}{\Longleftrightarrow}$ maximal (diagonalizable) abelian subalg of $\mathfrak{g}$

For $\alpha \in \mathfrak{t}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$, we set

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H) X \quad (H \in \mathfrak{t}) \}$$

Then $\mathfrak{g}$ has root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \underset{\substack{\uparrow \\ \text{root system for } (\mathfrak{t}, \mathfrak{g})}}{\Delta} = \{ \alpha \in \mathfrak{t}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0 \}$$

<u>Fact</u>
$$\begin{cases} \cdot\ \alpha \in \Delta \Rightarrow -\alpha \in \Delta \\ \cdot\ \dim \mathfrak{g}_\alpha = 1 \end{cases}$$

Fix a subset of Δ s.t.

$$\begin{cases} \cdot\ \Delta = \Delta^+ \amalg (-\Delta^+) \\ \cdot\ \alpha, \beta \in \Delta^+,\ \alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in \Delta^+ \end{cases}$$

There exists $\exists H_\alpha$, $\exists E_{\pm\alpha}$ $(\alpha \in \Delta^+)$ s.t.

$$\begin{cases} \cdot\ \mathfrak{t} = \sum_{\alpha \in \Delta^+} \mathbb{C} H_\alpha, \quad \mathfrak{g}_{\pm\alpha} = \mathbb{C} E_{\pm\alpha} \\ \cdot\ \mathfrak{sl}_\alpha = \mathbb{C} H_\alpha + \mathbb{C} E_{+\alpha} + \mathbb{C} E_{-\alpha} \text{ is isom to } \mathfrak{sl}(2, \mathbb{C}) \\ \quad \text{via } H_\alpha \longleftrightarrow H_0,\ E_{\pm\alpha} \longleftrightarrow E_\pm \end{cases}$$

$$\rightarrow \mathfrak{g} = \sum_{\alpha \in \Delta^+} \mathfrak{sl}_\alpha$$

<u>Rem</u> When $\mathfrak{g}$ is simple, $\quad n = 6, 7, 8$

type of Δ, $\underset{\substack{\uparrow \\ \mathfrak{sl}(n+1, \mathbb{C})}}{A_n}$, B_n, $\underset{\substack{\uparrow \\ \mathfrak{sp}(2n, \mathbb{C})}}{C_n}$, D_n, E_n, F_4, G_2

Ex Let $E_{ij} = \begin{array}{cc} & \\ i & \begin{pmatrix} \cdots & 1 & \\ & \vdots & \end{pmatrix} \\ & \,\, j \end{array}$

① $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$

$$\mathfrak{t} = \left\{ H = \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{C}) \right\}$$

$e_i \in \mathfrak{t}^*$ def by $e_i(H) = h_i$

$\Delta^+ = \{ e_i - e_j \mid 1 \le i < j \le n \}$

$H_{e_i - e_j} = E_{ii} - E_{jj}$ $\quad E_{e_i - e_j} = E_{ij}$ $\quad E_{-\alpha} = {}^t E_\alpha$
$(\alpha \in \Delta^+)$

② $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$

$$\mathfrak{t} = \left\{ H = \left(\begin{array}{ccc|ccc} h_1 & & & & & \\ & \ddots & & & & \\ & & h_n & & & \\ \hline & & & -h_1 & & \\ & & & & \ddots & \\ & & & & & -h_n \end{array} \right) \middle| \ h_i \in \mathbb{C} \right\}$$

$e_i \in \mathfrak{t}^*$ def. by $e_i(H) = h_i$

$\Delta^+ = \{ e_i + e_j \,;\, e_i - e_j \ \ (1 \le i < j \le n) \}$
$\qquad \cup \{ 2e_k \mid 1 \le k \le n \}$

$$H_{e_i \pm e_j} = \left(\begin{array}{c|c} E_{ii} \pm E_{jj} & \\ \hline & -(E_{ii} \pm E_{jj}) \end{array} \right), \quad H_{2e_k} = \left(\begin{array}{c|c} E_{kk} & \\ \hline & -E_{kk} \end{array} \right)$$

$$E_{e_i + e_j} = \left(\begin{array}{c|c} & E_{ij} + E_{ji} \\ \hline & \end{array} \right), \quad E_{e_i - e_j} = \left(\begin{array}{c|c} E_{ij} & \\ \hline & -E_{ji} \end{array} \right)$$

$$E_{2e_k} = \left(\begin{array}{c|c} & E_{kk} \\ \hline & \end{array} \right), \quad E_{-\alpha} = {}^t E_\alpha \quad (\alpha \in \Delta^+)$$

§6. Theorem of highest weight

$\Gamma = \{\gamma \in \mathfrak{t}^* \mid \gamma(H_\alpha) \in \mathbb{Z} \quad (\alpha \in \Delta^+)\}$

$\Lambda = \{\lambda \in \Gamma \mid \lambda(H_\alpha) \geq 0 \quad (\alpha \in \Delta^+)\}$

$\quad = \{\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$

if $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C}),\ \mathfrak{sp}(2n, \mathbb{C})$

(π, V) : fin. dim. $\mathbb{C}$-lin rep of $\mathfrak{g}$

Since $\pi|_{\mathfrak{sl}_\alpha} \simeq \bigoplus_i d\Phi_{l_i}$ and $\mathfrak{t}$ is abelian.

we have $V = \bigoplus_{\gamma \in \Gamma} V(\gamma)$

$V(\gamma) := \{v \in V \mid \pi(H)v = \gamma(H)v \ (H \in \mathfrak{t})\}$

γ weight of $\pi \iff \gamma \in \Gamma$ s.t $V(\gamma) \neq 0$

<u>Theorem</u> (thm of highest weight)

(π, V) irred. fin. dim $\mathbb{C}$-lin. rep of $\mathfrak{g}$

Then $0 \neq {}^\exists v_0 \in V$ (unique up to scalar) s.t.

$v_0 \in V(\lambda_\pi)$ $({}^\exists \lambda_\pi \in \Gamma)$, $\pi(E_\alpha)v = 0$ $(\alpha \in \Delta^+)$

(λ_π is called the highest wt of π)

Moreover,

$$\{\text{irred. finite dim } \mathbb{C}\text{-lin rep of } \mathfrak{g}\}/\simeq \ \longrightarrow \Lambda \text{ is bijective}$$

$$\pi \longmapsto \lambda_\pi$$

Thm (Borel–Weil Thm)

$T = \exp(\mathfrak{t})$, $N_- = \exp\left(\sum_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}\right)$

For $\lambda \in \Lambda$, we define char ξ_λ of T

by $\xi_\lambda(\exp(H)) = e^{\lambda(H)}$ $(H \in \mathfrak{t})$

$$W_\lambda := \left\{ \underset{\text{holomorphic}}{F : G \to \mathbb{C}} \;\middle|\; \begin{array}{l} F(ntg) = \xi_\lambda(t) F(g) \\ (n,t,g) \in N_- \times T \times G \end{array} \right\}$$

$$(\Pi_\lambda(g)F)(x) = F(xg) \text{ for } g \in G \;\; F \in W_\lambda$$

Then (Π_λ, W_λ) is irred. fin. dim. hol. rep of G

with ht wt λ

§7. computable realization of hol. rep of $Sp(4,\mathbb{C})$

$Sp(4,\mathbb{C}) \curvearrowright \mathbb{C}^4$ by matrix multiplication

$\xrightarrow{\text{induce}}$ the action $\hat{\Phi}$ on $R := Sym(\mathbb{C}^4) \otimes Sym(\Lambda^2\mathbb{C}^4)$

$\left(Sym(V) = \bigoplus_{\ell \geq 0} Sym^\ell(V) \right.$: sym alg of V $\left.\right)$

$\{e_i\}_{i=1}^4$ basis of $\mathbb{C}^4$ def. by $(e_1, e_2, e_3, e_4) = 1_4$

$\{e_{ij}\}_{1\leq i<j\leq 4}$ basis of $\Lambda^2\mathbb{C}^4$ defined by $e_{ij} := e_i \wedge e_j$

$$R^\lambda := Sym^{\lambda_1-\lambda_2}(\mathbb{C}^4) \otimes Sym^{\lambda_2}(\Lambda^2\mathbb{C}^4)$$

for $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \in \Lambda$

I_R : ideal of R gen. by

$1\otimes(e_{13} + e_{24})$, $1\otimes(e_{12}e_{34} - e_{13}e_{24} + e_{14}e_{23})$

$e_i \otimes e_{jk} - e_j \otimes e_{ik} + e_k \otimes e_{ji}$ $(1 \leq i < j < k \leq 4)$

$\to R^\lambda$ and I_R are G-inv.

$(\Phi_\lambda, V_\lambda)$ quot. rep of $\hat{\Phi}|_{R^\lambda}$

on $V_\lambda = R^\lambda / (R^\lambda \cap I_R)$

$\to \Phi_\lambda$ is irred fin. dim hol rep of $Sp(4, \mathbb{C})$ with ht wt λ.

(Ref) Knappの緑本 赤本)