$GSp(4)$ から $GL(4)$ への generic transfer の解説 ①

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$G = GSp(4)_{/F}$ F : number field

$= \{ g \in GL(4) \mid {}^{\exists} \nu(g) \in \mathbb{G}_m \text{ s.t. } {}^t g J g = \nu(g) J \}$

spin : $\hat{G} \cong GSp(4, \mathbb{C}) \hookrightarrow GL(4, \mathbb{C})$ ← the dual gp $GL(4)$

$G \ \pi \quad GL(4) \ \Pi$

$\pi = \bigotimes_v \pi_v$: auto. rep of $G_{\mathbb{A}}$ with unitary central char.

$\omega_\pi : Z_{\mathbb{A}} \cong \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$

「$v \notin S_\pi \Rightarrow v < \infty$ & $\pi_v^{K_v} \neq \{0\}$」 $\to A_v \in \hat{G}$

$\# S_\pi$ $\qquad K_v = GSp(4, F_v) \cap GL(4, \mathcal{O}_v)$

Principle of Functoriality predicts :

<u>Conj A</u> $\quad v \notin S_\pi$

$\pi_v'' :=$ irred adm. rep of $GL(4, F_v)$ with

Satake parameter $\mathrm{spin}(A_v) \in GL(4, \mathbb{C})$

$\Rightarrow {}^{\exists} \Pi = \bigotimes_v \Pi_v$: auto. rep of $GL(4)_{\mathbb{A}}$

s.t. $\Pi_v = \pi_v'' \quad ({}^\forall v \notin S_\pi)$

$\left(\Rightarrow L^{S_\pi}(s, \pi, \mathrm{spin}) = L^{S_\pi}(s, \Pi, \mathrm{std}_4) \right)$

$\mathrm{id} = \mathrm{Std}_m : GL(m, \mathbb{C}) \longrightarrow GL(m, \mathbb{C})$

Strategies

① Converse thm

Asgari - Shahidi 2008 Duke J Compositio

② theta correspondence

$$GSp(4) \longleftrightarrow GO(3,3)$$

(Yamana's talk)

Gan - Takeda preprint (to appear in Ann. of Math)

③ trace formula

Thm B = converse theorem (Cogdell - Kim - PS - Shahidi)

$\Pi' = \otimes \Pi'_v$: irred. adm. rep. of $GL_n(\mathbb{A})$

s.t. $\begin{cases} \omega_\Pi : Z_{\mathbb{A}} \longrightarrow \mathbb{C}^\times \text{ is trivial on } F^\times \\ L(s, \Pi', Std_n) \text{ converges absolutely} \end{cases}$

S : finite set of finite places of F

$$\mathcal{T}^S = \bigcup_{1 \le m \le n-1} \mathcal{T}^S(m)$$

$$= \bigsqcup_{1 \le m \le n-1} \left\{ \tau = \otimes \tau_v \;\middle|\; \begin{array}{l} \text{cusp. autom rep} \\ \text{of } GL(m)_{\mathbb{A}} \\ v \in S \Rightarrow \tau_v \text{ unram.} \end{array} \right\}$$

Assume $\exists \eta : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$, $\exists S$

$\forall \tau \in \mathcal{T}^S$

s.t. (E) $L(s, \Pi' \times (\tau \otimes \eta), std_n \otimes std_m)$ is entire fct.

(F.E.) $L(s, \Pi' \times (\tau \otimes \eta), \square)$

$$= \varepsilon(s, \Pi' \times (\tau \otimes \eta), \square) \times L(1-s, \Pi' \times (\tau \otimes \eta), \square^{\vee})$$

(BVS) $L(s, \Pi' \times (\tau \otimes \eta), \square)$

is bounded on $\{t_1 < \mathrm{Re}(s) < t_2\}$ $\quad {}^{\forall}(t_1, t_2)$

$\Longrightarrow {}^{\exists}\Pi = \bigotimes_v \Pi_v$: irred. auto. rep of $GL(n)_{\mathbb{A}}$

s.t $\Pi'_v \cong \Pi_v \quad (v \notin S)$

※ L-ft … J-PS-S. Shahidi

How to use conv. thm (= thm B).

$\pi = \otimes \pi_v \rightsquigarrow \Pi'$: provisonal candidate

$GSp(4)$ $\qquad GL(4)$

(not necessarilly automorphic)

$\underset{\text{Thm}}{\longrightarrow} {}^{\exists}\Pi$: auto rep. of $GL(4)$

satisfying Conj A

(i) $v \notin S_{\pi}$ $\quad \Pi'_v \overset{\text{def}}{:=} \Pi''_v$

(ii) $v | \infty$

$$\begin{array}{ccc} W_{F_v} & \xrightarrow{\phi_v} & GSp(4, \mathbb{C}) \\ & \searrow^{\tilde{\phi}_v} & \cap \\ & & GL(4, \mathbb{C}) \end{array}$$

$\phi_v \leftarrow \pi_v$

$\Pi'_v \leftarrow \tilde{\phi}_v$

Π'_v irred rep of $GL(4, F_v)$

(iii') $v \in S_\pi$ $v < \infty$

Take an arbitrary irred. admissible rep.

Π_v' of $GL(4, F_v)$

s.t. $\omega_{\Pi_v'} = \omega_{\pi_v}^2$

Note that $\bigotimes_v \omega_{\pi_v} = \omega_{\pi_2}^2$

Need to verify the assumptions in thm B.

· η : suff ramified $m = 1, 2, 3$

$$L(s, \Pi' \times (\tau \otimes \eta), Std_4 \otimes Std_m)$$

$$= L^{S_\pi}(s, \Pi' \times (\tau \otimes \eta), Spin_4 \otimes Std_m)$$

contribution

$$= L^{S_\pi}(s, \pi \times (\tau \otimes \eta), spin \otimes std_m)$$

$$= L(s, \pi \times (\tau \otimes \eta), spin \otimes std_m)$$

η : suff. ramified.

Want to prove the entireness of

<u>two possible method</u>

[A-S] $\begin{cases} \text{(a) zeta integral} \\ \text{(b) Langlands-Shahidi method} \end{cases}$

<u>Thm C ([A-S])</u>

Conj A is true if

π : cuspidal &

π has global Whittaker model (= globally generic)

$$\begin{array}{ccc} GSp(4) \times GL(m) & \subset & GSpin_{5+2m} \\ \downarrow & & \downarrow \\ SO_5 \times GL_m & \subset & SO_{5+2m} \\ \rotatebox{90}{$\in$} & & \\ (g, h) & \mapsto & \begin{pmatrix} \boxed{h} & & \\ & \boxed{g} & \\ & & \boxed{{}^t h^{-1}} \end{pmatrix} \end{array}$$

$$\left.\begin{array}{l} \varphi_\pi \in \pi \\ \varphi_\tau \in \tau \end{array}\right\} \text{cusp forms}$$

$\rightsquigarrow E(s, \varphi_\pi, \varphi_\tau, x)$ Eisenstein series on $GSpin_{5+2m}$

$$E_\psi(s, \varphi_\pi, \varphi_\tau, x) := \int_{U_F \backslash U_{\mathbb{A}}} E(s, \varphi_\pi, \varphi_\tau, ux) \psi(u)^{-1} du$$

$$\begin{array}{ccc} U & \subset & GSpin_{5+2m} \\ \downarrow & & \\ U & \subset & SO_{5+2m} \\ \rotatebox{90}{$\in$} & & \end{array}$$

$$\begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \longmapsto \psi\left(\sum_{i=1}^{5+2m-1} x_{i,i+1}\right) \in \mathbb{C}^{(1)}$$

$$\psi : \mathbb{A}/F \longrightarrow \mathbb{C}^{(1)}$$

By using this we can prove (F), (FE) (BVS

$$E_\psi(s, \varphi_\pi, \varphi_\tau, e) = \prod_{v \in} (\text{local integral}) \times L^{S_\pi}(1+s, \pi \times \tau, \mathrm{spin} \boxtimes \mathrm{Std}_m)^{-1} \times L^{S_\pi}(1+2s, \tau, \mathrm{Sym}^2)^{-1}$$

By using this, we can prove (E), (FE), (BVS)

<u>two possibilities</u>

① Π is cuspidal

② $L^S(s, \pi, \mathrm{std}_4) = L^S(s, \pi_1, \mathrm{std}_2) \times L^S(s, \pi_2, \mathrm{std}_2)$