森山知則

Kohnen-Skoruppa の論文の紹介

(2010/8/4)　　Invent. Math. '89

- Spinor L-fct of $GSp(4)$

① zeta integrals

1-a　Andrianov's integral (previous talk)

1-b　Novodvorsky's integral ('75 C.R. Paris)

1-c　Kohnen-Skoruppa ('79 PSPM 33-2)

1-d　Murase-Sugano ('95 Math. Ann)

② Langlands-Shahidi (1980 〜)

(yesterday's talk)　$GSp(4) \times GL(1) \subset GSpin$

$F \in S_k(Sp(2,\mathbb{Z}))$: Hecke eigen.

$$\overset{\shortparallel}{\sum_{Q>0}} A(Q)\, e(\mathrm{tr}(QZ))$$

$$\rightsquigarrow Z_F^*(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

$$Z_F(s) = \prod_{p<\infty} \det\left(I_4 - A_p p^{-s+k-3/2}\right)^{-1}$$

$$= \prod_{p<\infty}\Big(1-\lambda_F(p)p^{-s} + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})p^{-2s}$$

$$- \lambda_F(p)p^{2k-3-3s} + p^{4k-6-4s}\Big)^{-1}$$

spin zeta fct of F.

- A_p : Satake parameter of F $\in GSp(4,\mathbb{C})$
- $\lambda_F(n)$ eigen value of $T(n)$ on F.

1-a: An $\longleftarrow$ Bessel (= Generalized Whittaker model)

1-b: No $\longleftarrow$ Whittaker model

1-c: Koh-Sk $\longleftarrow$ Fourier-Jacobi exp.

1-d: Mu-Su $\longleftarrow$ Shintani model

<u>Thm A</u> $\quad {}^{\exists}Q>0$ s.t $A(Q)\neq 0$ Q represents 1

$\Rightarrow$ (i) $Z_F^*(s)$ is continued to a mero. fct. on $\mathbb{C}$.

(ii) $Z_F^*(s) = Z_F^*(2k-2-s)$

(iii) $Z_F^*(s)$ is entire

$\Uparrow$ (Evdokimov, Oda). $\quad \Longleftrightarrow F \perp S_k^*(Sp(2,\mathbb{Z}))$

$$S_k^*(Sp(2,\mathbb{Z})) \subset S_k(Sp(2,\mathbb{Z}))$$

<u>注</u> $\quad F \in S_k^*(Sp(2,\mathbb{Z}))$

$\Longleftrightarrow F(z) = \sum_{Q>0} A(Q)\, e(\mathrm{Tr}\, Qz)$

$$A\left(\begin{pmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{pmatrix}\right) = \sum_{\delta \mid \gcd(\alpha,\beta,\gamma)} \delta^{k-1} A\begin{pmatrix} \alpha\gamma/\delta^2 & \beta/2\delta \\ \beta/2\delta & 1 \end{pmatrix}$$

$\Longleftrightarrow$ $A(Q)$ depends only on $\mathrm{disc}(Q) = -\det(2Q)$ & $\gcd(\alpha, \beta, \gamma)$

$F \in S_k^*(Sp(2,\mathbb{Z}))$ Hecke-eigen.

$\Leftrightarrow$ (S) $= \zeta(s-k+1)\zeta(s-k+2)L(s, \exists g)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cap$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad S_{2k-2}(SL_2(\mathbb{Z}))$

Saito-Kurokawa corresp.

$\Leftrightarrow \pi_{F,p} \simeq \mathrm{Ind}_{P_{S,p}}^{G_p}(\pi_{g,p} \otimes \delta_p^{1/2}) \quad (\forall p)$

Siegel Parabolic

Proof of Theorem A

Step 1 certain Rankin-Selberg integral

$$\mathfrak{h}_2 \ni \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \qquad \tau' \mapsto \tau' + 1$$

$F, G \in S_k(Sp(2,\mathbb{Z}))$

$$F(Z) = \sum_{N=1}^{\infty} \phi_N(\tau, z) e(N\tau')$$

$$G(Z) = \sum_{N=1}^{\infty} \psi_N(\tau, z) e(N\tau')$$

$\phi_N, \psi_N \in J_{k,N}^{cusp}$ the space of Jacobi forms of wt k index N

$$D_{F,G}(s) = \zeta(2s-2k+4) \times \sum_{N=1}^{\infty} \langle \phi_N, \psi_N \rangle N^{-s} \quad \left(\begin{matrix}\text{abs. conv} \\ \mathrm{Re}(s) > k+1\end{matrix}\right)$$

Prop B ① $D_{F,G}$ is continued to a mero. fct on $\mathbb{C}$

② $D_{F,G}(s)$ is holo. except for a possible simple pole at $s=k$. $\mathrm{Res}_{s=k} = \mathrm{const}\langle F, G\rangle$

③ $$D^*_{F,G}(s) \overset{def}{:=} (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)D_{F,G}(s)$$

$$= D^*_{F,G}(2k-2-s)$$

proof of Prop B

$$P_J = \left\{ \left(\begin{array}{cc|cc} a & 0 & b & * \\ * & a_2 & * & * \\ \hline c & 0 & d & * \\ 0 & 0 & 0 & a_2^{-1} \end{array}\right) \in Sp(2) \right\}$$

$$E: \mathrm{Ind}_{P_{J,\mathbb{A}}}^{G_{\mathbb{A}}}\left(\mathbb{C}\otimes|a_2|_{\mathbb{A}}^s\right) \hookrightarrow \mathcal{A}(Sp(2,\mathbb{A}))$$

$\phi \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix}$

(c.f. $g \in \mathcal{A}_0(GL(2))$
π_g Klingen-Eisenstein series)

Eisenstein series

$$E_s(z) := \sum_{M\in P^0_{J,\mathbb{R}}\cap Sp(2,\mathbb{Z})\backslash Sp(2,\mathbb{Z})} \left(\frac{\det(\mathrm{Im}(M\langle z\rangle))}{\mathrm{Im}(M\langle z\rangle_1)}\right)^s$$

$$M\langle z\rangle = \begin{pmatrix} M\langle z\rangle_1 & * \\ * & * \end{pmatrix}$$

Fact • $E_s^*(Z) \overset{\text{def}}{:=} \zeta_{\mathbb{Q}}(2s) E_s(Z)$

$$= E_{2-s}^*(Z)$$

• poles $\begin{cases} s=0 & \mathrm{Res}_{s=0} = 1 \\ s=2 & \mathrm{Res}_{s=2} = -1 \\ s \neq 0, 2 & \text{holom.} \end{cases}$

Prop C $\displaystyle\int_{Sp(2,\mathbb{Z})\backslash \mathfrak{H}_2} F(Z)\overline{G(Z)} \underset{wt=0}{E_s^*(Z)} d^*Z$

$$\overset{\text{unfold}}{=} \cdots = \pi^{2-k} D_{F,G}^*(s)$$

Thm B follows from Prop C, Fact on $E_s^*(Z)$

Step 2

$F \in S_k(Sp(2,\mathbb{Z}))$: Hecke eigen

Prop D $J_{k,1}^{cusp} \xrightarrow{\sim} S_k^*(Sp(2,\mathbb{Z}))$

$$\phi \longmapsto \sum_{N=1}^{\infty} (V_N \phi)(\tau, z)\, e(N\tau')$$

$\exists V_N : J_{k,1}^{cusp} \longrightarrow J_{k,N}^{cusp}$

$V_N^* : J_{k,N}^{cusp} \longrightarrow J_{k,1}^{cusp}$

adjoint of V_N (w.r.t. Peterson inner prod.)

choice of $G \in S_k^*(Sp(2,\mathbb{Z}))$

$$J_{k,1}^{cusp} \ni \phi(\tau,z) = \sum_{D<0} \sum_{\substack{r\in\mathbb{Z} \\ D\equiv r^2 (4)}} C(D,r)\, e\left(\frac{r^2-D}{4}\tau + rz\right)$$

$\exists !\ P_{k,D,r} \in J_{k,1}^{cusp}$

s.t. $\langle \phi, P_{k,D,r} \rangle = C(D,r)$

$\mathcal{P}_{k,D,r}(Z) \in S_k^*(Sp(2,\mathbb{Z}))$

For simplicity $D=-4$

$$D_{F,P_{k,D,r}}(s) \times \zeta(2s-2k+4)^{-1}$$

$$= \sum_{N=1}^{\infty} \langle \phi_N, V_N P_{k,D,r} \rangle N^{-s}$$

$$= \sum_{N=1}^{\infty} \langle \underbrace{V_N^* \phi_N}_{\exists \text{ explicit form}}, P_{k,D,r} \rangle N^{-s}$$

$$= \zeta_{\mathbb{Q}(\sqrt{-1})}(s-k+2) \times \sum_{N=1} A\left(\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}\right) N^{-s}$$

$$= A\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) Z_F(s)$$

(Andrianov 1970)

If $A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$.

F, G, E_s^*
$\cap$
$S_k^*(Sp(2,\mathbb{Z}))$

$I_{P_S}^G(\pi_g)$

$I_{P_J}^G(\sigma \times |q_2|^s)$

Thm A follows from

Thm B & (☆)

Bump-D-D