山下剛

局所体上のGL(2)の既約許容的表現概説

(2010/8/3)

Plan

§0. Basic notions
(smooth rep's, .., L-factors, ε-factors)

§1. Recovery of the representation from L & ε-factors

local ↑

§2. local Langlands correspondences for $p>2$.
$\pi_v \rightsquigarrow \sigma_v$

↓ global

§3. global-local compatibilities

§-1, G : finite gp

$$\begin{cases} \cdot\ \#G = \displaystyle\sum_{\pi:\text{ irred. rep of } G} (\dim \pi)^2 \\ \cdot\ \#\text{ conj classes of } G = \#\text{ irred. rep of } G/\cong \end{cases}$$

$G = GL_2(\mathbb{F}_q)$ $\quad \# GL_2(\mathbb{F}_q) = (q^2-1)(q^2-q)$

conj classes of $G \rightsquigarrow$ irred. rep of G

"parametrized by dual"

① $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $\quad (a \neq 0) \quad \# = q-1$

② $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ $\quad (a \neq 0) \quad \# = q-1$

③ $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $\quad (a \neq b) \quad \# = \dfrac{(q-1)(q-2)}{2}$

$p>2$
$\varepsilon \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$

④ $\mathbb{F}_{q^2}^\times \hookrightarrow GL_2(\mathbb{F}_q)$ $\quad \# = \dfrac{q^2-q}{2}$ $\quad \varphi = x+\sqrt{\varepsilon}\, y \mapsto \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$,

$\longleftrightarrow \chi \circ \det \quad \chi : \mathbb{F}_q^\times \longrightarrow \mathbb{C}^\times \quad \dim = 1 \quad \# = q-1$

$\longleftrightarrow \chi \circ \det \otimes St \quad \chi : \mathbb{F}_q^\times \to \mathbb{C}^\times \quad \dim = 1 \quad \# = q-1$

$\longleftrightarrow \mathrm{Ind}_B^G(\chi_1, \chi_2) \quad \chi_1, \chi_2 : \mathbb{F}_q^\times \longrightarrow \mathbb{C}^\times \quad \# = \frac{(q-1)(q-2)}{2}$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad \chi_1 \neq \chi_2 \quad \dim = q+1$

$\Uparrow$

$\# G/B = q+1 \qquad \mathrm{Ind}(\chi_1, \chi_2) = \mathrm{Ind}(\chi_2, \chi_1)$

$G/B \cong \mathbb{P}^1(\mathbb{F}_q)$

$\longleftrightarrow \Theta_\varphi \quad \varphi : \mathbb{F}_{q^2}^\times \longrightarrow \mathbb{C}^\times$ char $\quad \# = \frac{q^2-q}{2}, \quad \dim = q-1$

cuspidal ← Weil rep

φ not factors through $\mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$

$F : \Theta_\varphi \longrightarrow \mathbb{C}$

$F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v\right) = \psi(ax) F(v)$

$$(q-1)\cdot 1^2 + (q-1)q^2 + \frac{(q-1)(q-2)}{2}(q+1)^2 + \frac{q^2-q}{2}(q-1)^2$$

$$= (q^2-1)(q^2-q)$$

Weil rep of $GL_2(\mathbb{F}_q)$

$F = \mathbb{F}_q, \quad E = \begin{cases} F \oplus F \\ \text{quad. ext. of } F \end{cases}$

$(a, b) \in E \quad \overline{(a, b)} = (b, a)$

$a \in E, \ \bar{a} = $ conj of a over F

$\mathrm{tr}(a) = a + \bar{a}, \quad N(a) = a\bar{a}$

$S(E) := \{ f(\text{ctions}) : E \to \mathbb{C} \}$ $\quad \dim = q^2$

$1 \neq \psi : F \to \mathbb{C}^\times$ char.

$SL_2(F) \curvearrowright S(E)$ action defined as follows

$\Phi \in S(E)$

$$\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \Phi \right)(v) = \Phi(av)$$

$$\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \Phi \right)(v) = \psi(xNv)\, \Phi(v)$$

$$\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi \right)(v) = \gamma_E\, \hat{\Phi}(v)$$

$$\gamma_E = \begin{cases} +1 & E = F \oplus F \\ -1 & \text{quad. ext.} \end{cases}$$

$$\hat{\Phi}(u) := \frac{1}{q} \sum_{v \in E} \Phi(v)\, \psi(\mathrm{tr}(\bar{v} \cdot u))$$

<u>Prop</u> this is well-defined.

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2 x \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -x^{-1} & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}$$

$\varphi : E^\times \longrightarrow \mathbb{C}^\times$, $E^\times \searrow F^\times$

not factors through.

$|E^\times|_1 = \{ a \in E^\times \mid Na = 1 \}$

$\curvearrowright S(E)(\varphi)$

$SL_2(F) := \{ \Phi \in S(E)$

$$\dim = \begin{cases} q+1 & \text{split} \\ q-1 & \text{non split} \end{cases}$$

extend the action $SL_2(F) \curvearrowright S(E)(\varphi)$

to $GL_2(F) \curvearrowright S(E)(\varphi)$

as follows.

$a \in F^\times$, $NA = a$ $\quad A \in E^\times$ $(E^\times \twoheadrightarrow F^\times)$

$$\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Phi\right)(v) = \varphi(A)\,\Phi(Av)$$

well defined $GL_2(F) \curvearrowright S(E)(\varphi)$

irred.

<u>Fact</u> $GL_2(F) \curvearrowright S(E)(\varphi)$

$$\begin{cases} \cong \pi_1(\chi_1, \chi_2) & \text{if } E = F \oplus F \\ \text{cuspidal} & \text{if } E \text{ quad ext. of } F \end{cases}$$

$$S(E)(\varphi) \cong \pi(\chi_1, \chi_2)$$

$$\Phi \longmapsto \left(g \mapsto (g\Phi)(0,1)\right)$$

<u>modifications for local fields</u>

① $S(E) = C_c^\infty(E)$

unitarize

② $\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \Phi\right)(v) := \|a\|(-1, a)\,\Phi(a, v)$

③ $E^\times \xrightarrow{N} F^\times$ not surj

$$GL_2(F)_+ = \{ g \in GL_2(F) \mid \det g \in \operatorname{Im} N \}$$

$$\operatorname{Ind}_{GL_2(F)_+}^{GL_2(F)} S(E)(\varphi) =: \omega_\varphi$$

cf. $E \rightsquigarrow D$ quaternion $/F$

$d : D^\times \curvearrowright W$ finite irred. smooth rep.

$|D^\times|_1 := (D^\times)^{Nrd=1}$

$(S(D) \otimes W)(d)$

$:= \left\{ \begin{array}{l} \Phi \in S(D) \otimes W \\ \Phi(vg) = d(g)^{-1} \Phi(\bar{v}) \text{ for } \forall g \in |D^\times|_1 \end{array} \right\}$

$\dim d = 1 \Rightarrow \Theta_d$ special

$\dim d > 1 \Rightarrow \Theta_d$ super cuspidal

F : non Archimedian local field $G := GL_2(F)$

$q = \#$ of residue field.

$\|a\| := q^{-v(a)}$ $(v(\varpi_K) = 1)$ for $a \in F$

V : $\mathbb{C}$-vect space

$\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ gp hom

__Def__ (π, V) is called <u>smooth</u>

$\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ $\forall v \in V$, $\mathrm{Stab}_G(v) < G$ open

$\Longleftrightarrow V = \bigcup_{\substack{G \supset K \\ \text{open}}} V^K$

(π, V) smooth rep <u>admissible</u> $\Longleftrightarrow \forall K \subset G$ open $\dim V^K < \infty$

N.B.

$\forall$ irred. smooth are admissible

$G \supset H$ closed subgp

(σ, W) smooth rep of H

$$\mathrm{Ind}_H^G \sigma := \left\{ \begin{array}{l} f : G \to W \\ \text{s.t.} \cdot f(hg) = \sigma(h) f(g) \\ \qquad \text{for } \forall h \in H, g \in G \\ \quad \cdot \exists K \subset G \text{ open cpt} \\ \qquad \text{s.t. } f(gk) = f(g) \ \forall g \in G, \forall k \in K \end{array} \right\}$$

$G \curvearrowright$

$(gf)(x) := f(xg)$

smooth action.

$$\begin{array}{l} \mathrm{Ind}_H^G \sigma \\ \cup \\ c\text{-}\mathrm{Ind}_H^G \sigma := \left\{ \begin{array}{l} f \in \mathrm{Ind}_H^G \sigma \\ \cdot \text{the image of supp } f \text{ in } H \backslash G \\ \quad \text{cpt} \end{array} \right\} \end{array}$$

compact induction

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) = \mathrm{Hom}_H(\pi|_H, \sigma)$$

$$\mathrm{Hom}_G(c\text{-}\mathrm{Ind}_H^G \sigma, \pi) = \mathrm{Hom}_H(\sigma, \pi|_H)$$

$\pi \in$ smooth adm. rep of G

(π, V) smooth rep of G

$$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

$$V^\vee = \bigcup_{\substack{K \subset G \\ \text{open cpt}}} (V^*)^K$$

contragradient rep of V

smooth. $G \curvearrowright$

$$\langle \check{\pi}(g)\check{v}, v \rangle = \langle \check{v}, \pi(g^{-1})v \rangle$$

$$g \in G,\ v \in V,\ \check{v} \in \check{V}$$

<u>Prop</u> $V \to V^{\vee\vee}$ is isom $\Leftrightarrow$ V : admissible

<u>Prop</u> (π, V) admissible rep of G

(π, V) irred. $\Longleftrightarrow (\check{\pi}, \check{V})$ irred

<u>N.B.</u> For general G : locally profinite gp

this does not hold

e.g. $M = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$ mirabolic gp

$\theta : N \longrightarrow \mathbb{C}^\times$ char.

$c\text{-}\mathrm{Ind}_N^M \theta$: irred.

$$(c\text{-}\mathrm{Ind}_N^M \theta)^\vee \cong \mathrm{Ind}_N^M \theta^\vee \not\cong c\text{-}\mathrm{Ind}_N^M \theta^\vee$$

(not irred.)

Notation

π: smooth rep of G.

$$\chi\pi := \chi\circ\det\otimes\pi$$

$$n\text{-}\mathrm{Ind}_B^G\sigma := \mathrm{Ind}_B^G(\delta_B^{-1/2}\otimes\sigma)$$

$$B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \qquad \delta_B : B \to \mathbb{C}^\times$$

$$d\mu_B(xb) = \delta_B(b)\,d\mu_B(x)$$

(μ_B: left invariant Haar measure)

$$\delta_B\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}u\right) = \| t_2/t_1 \|$$

$$(n\text{-}\mathrm{Ind}_B^G\sigma)^\vee \cong n\text{-}\mathrm{Ind}_B^G(\sigma^\vee)$$

Jacquet module

$$G \supset B \supset N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \qquad B = T \ltimes N$$

Jacquet functor $\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(T)$

$$(\pi, V) \longrightarrow (\pi_N, V_N)$$

(V_N: N-coinvariant)

$$V/V(N) \qquad V(N) = \langle \sigma v - v \mid \sigma\in G,\ v\in V \rangle$$

$(\sigma, W) \in \mathrm{Rep}(T)$ extends to B

$$\sigma\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} := \sigma\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \qquad \pi:\text{ irred.}$$

$\Rightarrow$ Rough classification

π : irred. smooth rep'n of G.

$\Rightarrow$ π is one of the following

i) $\pi \cong n\text{-}\mathrm{Ind}_B^G(\chi_1, \chi_2)$

ii) $\pi \cong \chi \circ \det \qquad \chi : F^\times \to \mathbb{C}^\times$ char

iii) $\pi \cong \chi \, St_G$

$$\left(\begin{array}{l} 0 \longrightarrow 1_G \longrightarrow \mathrm{Ind}_B^G 1 \longrightarrow St_G \longrightarrow 0 \\ \rightsquigarrow 0 \longrightarrow \chi \circ \det \longrightarrow \mathrm{Ind}_B^G \chi \longrightarrow \chi St_\chi \longrightarrow 0 \end{array} \right)$$

iv) π : super cuspidal.

$$\pi(\chi_1, \chi_2) \simeq \pi(\chi_1', \chi_2')$$

$$\Leftrightarrow \{\chi_1, \chi_2\} = \{\chi_1', \chi_2'\}$$

$$\pi(\chi_1, \chi_2) \cong \pi(\chi_2, \chi_1)$$

$$\chi \, \pi(\chi_1, \chi_2) \cong \pi(\chi\chi_1, \chi\chi_2)$$

$$\pi(\chi_1, \chi_2)^\vee \cong \pi(\chi_1^{-1}, \chi_2^{-1})$$

$$(\chi \, St_G)^\vee \cong \chi^{-1} St_G$$

π : super cuspidal $\Leftrightarrow$ $\chi\pi$: super cuspidal.

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \sigma) \cong \mathrm{Hom}_B(\pi, \sigma)$$
$$\cong \mathrm{Hom}_T(\pi_N, \sigma)$$

$\rightsquigarrow V_N \neq 0 \iff \pi \hookrightarrow \mathrm{Ind}_B^G \chi$ $\exists$ character

<u>Thm</u> $\chi = (\chi_1, \chi_2) : T \to \mathbb{C}^\times$ char

(i) $n\text{-}\mathrm{Ind}_B^G \chi$: reducible $\iff \chi_1 \chi_2^{-1} = \| \ \|^{\pm 1}$

(ii) Assume $n\text{-}\mathrm{Ind}_B^G \chi$: reducible

a) G-composition length $= 2$

b) one is infinite dimensional, the other is one dimensional,

c) $n\text{-}\mathrm{Ind}_B^G \chi \supset \exists (1\text{-dim } G\text{-subgp})$
$\iff \chi_1 \chi_2^{-1} = \| \ \|^{-1}$

d) $n\text{-}\mathrm{Ind}_B^G \chi \twoheadrightarrow \exists (1\text{-dim } G\text{-subgp})$
$\iff \chi_1 \chi_2^{-1} = \| \ \|$

<u>Thm</u> π : irred. smooth rep. of G.
G : supercuspidal.

( ) $\forall v \in V, \forall \check{v} \in V^\vee$

$\gamma_{\check{v} \otimes v} : (G \ni g \longmapsto \langle \check{v}, gv \rangle \in \mathbb{C})$

matrix coeff are compactly suppoted
modulo $Z \subset G$ $\mathcal{C}(\pi) := \langle \gamma_{\check{v} \otimes v} \mid {v \in V \atop \check{v} \in V^\vee} \rangle$

L-factors & ε-factors

$1 \neq \psi : F \longrightarrow \mathbb{C}^\times$ additive char.

$$A = M_2(F)$$

$$\psi_A : \psi \circ tr_A : A \longrightarrow \mathbb{C}^\times$$

$\Phi \in C_c^\infty(A)$

Fourier transform $\hat{\Phi}(x) := \int_A \Phi(y) \psi_A(xy) d\mu_A(y)$

↑ Haar measure

Choose $d\mu_A$

so that

$\hat{\hat{\Phi}}(x) = \Phi(-x)$

$d\mu_\psi$ a self-dual measure

$$d\mu_\psi\left(\begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{O}_F & \mathcal{O}_F \end{pmatrix}\right) = q^{2l} \quad l = \min q \quad \ker \psi \supset \varpi^q$$

<u>Zeta integral</u>

$\Phi \in C_c^\infty(A)$, $f \in \mathcal{C}(\pi)$

$$\zeta(\Phi, f, s) = \int_G \Phi(x) f(x) \|\det x\|^s d\mu(x)$$

— (✲)

<u>Thm</u> (π, V) : smooth irred. rep. of G

(i) $\exists s_0 \in \mathbb{R}$, (✲) converges abs & uniform in vertical strips

(✡) is a rat. fct in q^{-s}

(ii) $Z(\pi) := \{ \zeta(\Phi, f, s+\frac{1}{2}) \mid \Phi \in C_c^\infty(A), f \in \mathcal{C}(\pi) \}$

$\Rightarrow$ $\exists !$ polynomial $P_\pi(X) \in \mathbb{C}[X]$ s.t

- $P_\pi(0) = 1$
- $Z(\pi) = P_\pi(q^{-s})^{-1} \mathbb{C}[q^s, q^{-s}]$

$f \in \mathcal{C}(\pi)$

$\check{f}(g) = f(g^{-1}) \in \mathcal{C}(\pi^\vee)$

<u>Thm (Funct. eq)</u>

(π, V) : irred. smooth rep. of G

$\Rightarrow$ $\exists$ rat. funct. $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$

s.t $\zeta(\hat{\Phi}, \check{f}, \frac{3}{2} - s) = \gamma(\pi, s, \psi) \zeta(\Phi, f, s + \frac{1}{2})$

for $\forall \Phi \in \mathcal{C}_c^\infty(A)$, $f \in \mathcal{C}$

$L(\pi, s) = P_\pi(q^{-s})$

⋮

<u>Cor</u> $\varepsilon(\pi, s, \psi)\varepsilon(\check{\pi}, 1-s, \psi) = \omega_\pi(-1)$

$\exists a \in \mathbb{C}^\times$, $\exists b \in \mathbb{Z}$ s.t.

$\varepsilon(\pi, s, \psi) = a\, q^{bs}$

$$L(\pi, s) = \sum_{i=1}^{\nu} \zeta(\Phi_i, f_i, s+\tfrac{1}{2})$$

$$\Rightarrow \varepsilon(\pi, s, \psi) = L(\pi, 1-s)^{-1} \sum \zeta(\check{\Phi}_i, \check{f}_i, \tfrac{3}{2}-s)$$

$$\cap$$

$$\mathbb{C}[q^{s}, q^{-s}]$$

$$\varepsilon(\pi, s, \psi) \in \mathbb{C}[q^{s}, q^{-s}]$$

Prop π : super cuspidal

$$\Rightarrow Z(\pi) = \mathbb{C}[X, X^{-1}]$$

$$\rightsquigarrow L(\pi, s)$$

Thm $(\chi_1, \chi_2) : T \longrightarrow \mathbb{C}^\times$

π : G-comp factor of n-$\mathrm{Ind}(\chi_1, \chi_2)$

$1 \neq \psi : F \to \mathbb{C}^\times$ char.

i) $\pi \not\cong \phi St_G \Rightarrow L(\pi, s) = L(\chi_1, s) L(\chi_2, s)$

$$\varepsilon(\pi, s, \psi) = \varepsilon(\chi_1, s, \psi)\,\varepsilon(\chi_2, s, \psi)$$

ii) $\pi \cong \phi St_G = L(\pi, s) = L(\phi, s+\tfrac{1}{2})$

$$\varepsilon(\pi, s, \psi) = -\varepsilon(\phi, s, \psi)$$

§1. Recovery of rep. from L & ε factors

Thm $1 \neq \psi : F \longrightarrow \mathbb{C}^\times$ char.

π_1, π_2 : irred. smooth. rep. of $GL_2(F)$

$$L(\chi, \pi_1, s) = L(\chi, \pi_2, s)$$

$$\& \quad \varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi)$$

for $\forall \chi : F^\times \longrightarrow \mathbb{C}^\times$

$$\Rightarrow \pi_1 \cong \pi_2$$

$\forall \chi,\ L(\chi\pi, s) = 1 \iff \pi$: super cuspidal

I) π_1, π_2 not super cuspidal case

π : non-super cuspidal

I-i) $\exists \chi\ L(\chi\pi, s)$ deg $= 2$ case

$\qquad\qquad\quad \| $

$\qquad L(\vartheta_1, s) L(\vartheta_2, s)\quad \vartheta_1, \vartheta_2$ unram.

I-i) a) $\vartheta_1\vartheta_2^{-1} \neq \| \ \|^{H} \Rightarrow \pi(\vartheta_1, \vartheta_2)$ irred.

(unram. principal series up to twist)

$\chi\pi$ having L-factor $= L(\vartheta_1, s) L(\vartheta_2, s)$

is only $\pi(\vartheta_1, \vartheta_2)$

$$\Rightarrow \pi \cong \pi(\chi^{-1}\vartheta_1, \chi^{-1}\vartheta_2)$$

I-i) b) $\vartheta_1\vartheta_2^{-1} = \| \ \|^{\pm 1}$

$$\Rightarrow L(\vartheta_1, s) L(\vartheta_2, s) = L(\vartheta, s - \tfrac{1}{2}) L(\vartheta, s + \tfrac{1}{2})$$

$\rightsquigarrow \chi\pi \cong \vartheta\circ\det \Rightarrow \pi \cong \chi^{-1}\vartheta\circ\det$

I-ii) $\nexists \chi$ s.t. $L(\chi\pi, s)$ deg $= 2$

$\exists \chi$ s.t. $L(\chi\pi, s)$ deg $= 1$

$\quad\quad\quad \| $

$L(\vartheta, s)$ $\quad \vartheta$: unram.

$\pi(\vartheta, \vartheta')$ $\quad \vartheta'$: ram.

or $\vartheta|\ |\ |^{-\frac{1}{2}} St$.

I-ii) b) $\nexists \vartheta$, ramified s.t.

$L(\vartheta, \chi\pi, s) \neq 1$

$\Rightarrow \chi\pi \cong \vartheta|\ |\ |^{-\frac{1}{2}} St_G$

$\Rightarrow \pi \cong \chi^{-1}\vartheta|\ |\ |^{-\frac{1}{2}} St_G$

II. Super cuspidal

$\{\varepsilon\}_X \rightsquigarrow \pi$

Kirillov model

$1 \neq \vartheta : N \longrightarrow \mathbb{C}^\times$

(π, V) : irred. smooth rep of G

$\dim \pi = \infty$

$\Rightarrow c\text{-}\mathrm{Ind}_N^M \vartheta \subseteq {}^{\exists!}V_{Kir} \subseteq \mathrm{Ind}_N^M \vartheta$

$\mathbb{C}$-vect space (c.L)

$\exists ! \; \pi_{Kir} : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V_{Kir})$ s.t.

$$\begin{cases} \pi_{Kir} \cong \pi \text{ as } G\text{-rep.} \\ \pi_{Kir}|_M \text{ subrep. of } \mathrm{Ind}_N^M \vartheta \end{cases}$$

$M = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$

π : super cuspidal $\Rightarrow$ $c\text{-}\mathrm{Ind}_N^M \vartheta = V_{Kir}$.

$$\begin{array}{ccc} c\text{-}\mathrm{Ind}_N^M \vartheta & \cong & C_c^\infty(F^\times) \\ \Psi & & \Psi \\ f & \longmapsto & f|_{\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}} \end{array}$$

$\Rightarrow \pi_{Kir} : G \curvearrowright C_c^\infty(F^\times)$ Kirillov model.

$\phi \in C_c^\infty(F^\times)$ $\quad \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \phi \right)(x) = \phi(ax)$

$$\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \phi \right)(x) = \vartheta(nx)\,\phi(x)$$

$$\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \phi \right)(x) = \omega_\pi(a)\,\phi(x)$$

$G = B \amalg BwN \qquad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\Rightarrow \left. \begin{array}{l} \omega_\pi \\ \text{the action of } w \end{array} \right\}$ recovery of π.

ψ : fix can recover ω_π from

$\{\varepsilon(\chi\pi, s, \psi)\}_\chi$ ↗ omit

cf. $\{\varepsilon(\chi, \pi, s, \psi)\}_\psi \rightsquigarrow$

$$\varepsilon(\pi, s, \psi(a\cdot)) = \omega_\pi(a)\|a\|^{2s-1}\varepsilon(\pi, s, \psi)$$

$\Rightarrow$ can recover π from the action of

$$w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

the action of w $\leftrightarrow$ fct. eq. $\leftrightarrow$ ε-factor.

$\chi : F^\times \to \mathbb{C}^\times$ char., $k \in \mathbb{Z}$

$$\xi_{\chi,k}(x) := \begin{cases} \chi(x) & \text{if } \operatorname{val}(x) = k \\ 0 & \text{else} \end{cases}$$

<u>Thm</u> $\pi_{Kir}(w)\xi_{\chi,k} = \varepsilon(\chi^{-1}\pi, \frac{1}{2}, \psi)\, \xi_{\chi^{-1}\omega_\pi, -n(\chi^{-1}\pi,\psi)-k}$

where $n(\chi^{-1}\pi, \psi) \in \mathbb{Z}$

$$\varepsilon(\chi^{-1}\pi, s, \psi) = q^{n(\chi^{-1}\pi,\psi)(\frac{1}{2}-s)}\varepsilon(\pi, \tfrac{1}{2}, \psi)$$

$\Rightarrow$ recovery of π

§2. local Langlands corr.

$$\mathcal{G}_2(F) = \left\{ \begin{array}{r} \text{2-dim'l Frob s-s.} \\ \text{Weil-Deligne rep} \\ \text{of } W_F \end{array} \right\}$$

$$\mathcal{A}_2(F) = \left\{ \begin{array}{c} \text{irred. smooth rep} \\ \text{of } GL_2(F) \end{array} \right\} \Big/ \sim$$

Thm (LLC)

$1 \neq {}^\forall \psi \; F \to \mathbb{C}^\times$

$\exists! \, \pi : \mathcal{G}_2(F) \to \mathcal{A}_2(F)$ bij

s.t. $\left\{ \begin{array}{l} L(\chi\pi(\rho), s) = L(\chi \otimes \rho, s) \\ \varepsilon(\chi\pi(\rho), s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi) \\ \text{for } \forall \chi : F^\times \to \mathbb{C}^\times \end{array} \right.$

$\mathcal{G}_2$		$\mathcal{A}_2$
irred	$\longleftrightarrow$	super cuspidal
indecomposable	$\longleftrightarrow$	discrete series
reducible indecomposable	$\longleftrightarrow$	special
decomposable	$\longleftrightarrow$	principal series

<u>Lemma</u>

$\rho : W_F \longrightarrow GL_n(\mathbb{C})$ irred. smooth.

$\Rightarrow \rho \cong \mathrm{Ind}_{W_E}^{W_F} {}^{\exists}\chi \qquad \chi : W_E \longrightarrow \mathbb{C}^\times$

In particular $p > 2$. $\Rightarrow \mathcal{G}_2 \ni \rho$ irred.

$\Rightarrow \rho \cong \mathrm{Ind}_{W_E}^{W_F} \chi$

construction of π $\quad p > 2$

$$\mathcal{G}_2 = \begin{cases} \chi_1 \oplus \chi_2 & N = 0 \\ \chi \|\cdot\| \oplus \chi & N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \mathrm{Ind}_{W_F}^{W_E} \chi & \end{cases}$$

- $\pi(\chi_1 \oplus \chi_2) = \pi(\chi_1, \chi_2) \quad \chi_1 \chi_2^{-1} \neq \| \ \|^{\pm 1}$
- $\pi((\chi \| \ \| \oplus \chi), N = 0) := \chi \circ \det$
- $\pi\left((\chi \| \ \| \oplus \chi), N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \chi \, St_G$
- $\pi(\mathrm{Ind}_{W_E}^{W_F} \chi) := \Theta_\chi$ Weil rep

$\Rightarrow$ L-factors & ε-factors