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Result of Harris-Taylor and Taylor-Yoshida which Chida-san needs for his talk

F^+ : tot real , F/F^+ : tot. imag. quad.

Π : cusp. irred auto. rep. of $GL_n(\mathbb{A}_F)$

($n=4$ for application)

s.t. (RA) Π_∞ regular algebraic

i.e. $H^\bullet(\mathfrak{g}, K, \Pi_\infty \otimes V^\vee) \neq 0$

$^\exists V$: an irred. alg rep. $/\mathbb{C}$ of $\mathrm{Res}_{F/\mathbb{Q}} GL_n$

(CSD) $\Pi^\vee \cong \Pi^c$ $\quad c \in \mathrm{Gal}(F/F^+)$, $c \neq 1$

(DS) $^\exists w_0 \nmid \infty$: place of F s.t.

Π_{w_0} : ess sq. integrable

Fix $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

Main statement

Then (1) $\exists\, \rho_{\pi,\iota} : Gal(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ (semi-simple)

s.t. $w \nmid \ell\infty$ place of F

$$\iota\left(WD\left(\rho_{\pi,\iota}|_{W_{F_w}}\right)^{Fr-ss}\right) \underset{LLC}{\longleftrightarrow} \Pi_w \otimes |\det|_w^{\frac{1-n}{2}}$$

(2) (RPC) Π_w : tempered $\forall w \nmid \infty$

(3) (Irr) $\exists\, w_0' \nmid \ell\infty$, $\Pi_{w_0'}$: ess. sq integrable

$\Rightarrow \rho_{\pi,\iota}$: irred.

(4) $w \nmid \ell\infty$, $\sigma \in W_{F_w}$ α : an eigenval of $\rho_{\pi,\iota}(\sigma)$

$\Rightarrow \alpha \in \overline{\mathbb{Q}}$, $\exists\, i \in \mathbb{Z}$

s.t $|\iota'(\alpha)| = q_w^{\frac{i}{2}}$, $\forall \iota' : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$

(5) (Pure) In (4), if π_w : unram. $\sigma = \mathrm{Frob}_w$

$\Rightarrow i = n-1$

(6) ($\ell = p$) $w | \ell \Rightarrow \rho_{\pi,\iota}$: potentially semi-stable at w

$w | \ell$, Π_w unram. $\Rightarrow \rho_{\pi,\iota}$: crystalline at w.

§2. Ingredient of proof

1. Reduced to the case

$F = F^+E$, $\exists\, E$ quad. imag

2. the geom. and coh. of some simple Shimura varieties
$\rightsquigarrow$ Main result of [HT]

3 A base change by Clozel-Labesse.

4. Weight monodromy conj

Today's weight.

1 × 2 ◎ 3 △ 4 ○

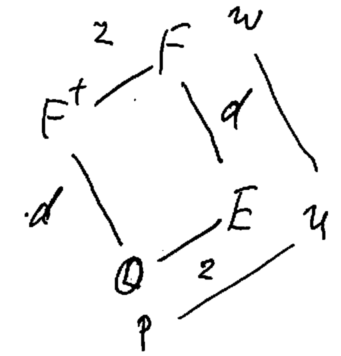
§3. Main result of [HT] - statement

§3.1 Setting

Fix p: a prime

Let $(E, u, F^+, d, F, w, \tau)$ be as follows;

- E : imagenary quad. field. s.t p splits
- $u | p$ a place of E (fix)
- F^+ : tot. real.
- $d = [F^+ : \mathbb{Q}]$
- $F = F^+ E$
- $w | u$ a place of F (fix)
- $\tau : F^+ \hookrightarrow \mathbb{R}$ (fix)



Let (A, B) as follows

$n \in \mathbb{Z}$, $n \geq 2$

· B a central div. alg $/F$

s.t. · $\dim B = n^2$

· $B^{op} \cong B \otimes_{F,c} F$

· v: place of F

$\Rightarrow B_v$ split or B_v : div alg

· $v = w$ or $v|_{F^+}$ not split over F

$\Rightarrow B_v$ split.

· If n : even

$\Rightarrow 1 + \frac{dn}{2} \equiv \frac{1}{2} \#\{v \mid B_v : \text{div. alg}\}$ (mod 2)

Fact

$\exists (*, B)$ satisfying

· $*$: positive involution of B

$* : B \to B$

- $(xy)^* = y^* x^*$

- $(x^*)^* = x$

· $\mathrm{Tr}_{B/\mathbb{Q}} (xx^*) > 0$ for $x \in B, x \neq 0$

s.t $*|_F = c$

· $\beta \in B$, $\beta \neq 0$ s.t.

· $\beta^* = -\beta$

· If we put

$G_\beta = \{(\lambda, g) \in \mathbb{G}_m \times (B^{op})^\times \mid g \cdot \underbrace{(\beta g^* \beta^{-1})}_{g^\#} = \lambda\}$

alg. gp $/\mathbb{Q}$

$G_{\beta,1} := \mathrm{Ker}\ G_\beta \longrightarrow \mathbb{G}_m \quad (\lambda, g) \mapsto \lambda$

then $G_{\beta,1}$: quasi-split at $\forall x \neq \infty$ place of $\mathbb{Q}$

x not split in $E/\mathbb{Q}$

(3) $V = B$ left B-module free of rk 1

$(\ ,\ ) : V \times V \longrightarrow \mathbb{Q}$ bilinear form

$(v, w) := \mathrm{tr}_{B/\mathbb{Q}}(v \beta w^*)$

$\Rightarrow \left(\begin{array}{l} \cdot\ (\ ,\ ) : \text{alternating} \\ -\ \text{Hermitian i.e.} \\ -\ (xv, w) = (v, x^* w) \end{array} \right)$

signature of $(\ ,\ ) \otimes_{\mathbb{Q}} \mathbb{R}$

is $\begin{cases} (1, n-1) & \text{at } \tau \\ (0, n) & \text{at } \tau' \neq \tau \end{cases}$

§3.2.
(Shimura var. of Harris–Taylor type)

$G = G_\tau = G_\beta$

$\rightsquigarrow \exists h : \underset{\substack{\shortparallel \\ \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m}}{\mathbb{S}} \longrightarrow G \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{R}$

- $h(\bar{z}) = \beta h(z)^* \beta^{-1}$
- $v, w \longmapsto \psi(v, h(\sqrt{-1})w)$ symm pos. definite

$\big(G(\mathbb{R})$-conj class of $h\big) = X \subset \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, \mathrm{Spec}(\mathbb{R}))$

is unique

(G, X) : Shimura datum

$\rightsquigarrow$ • $Sh(G,X) = (X_U)$ Shimura var.

$U \subset G(\mathbb{A}^\infty)$ compact open

- X_U : proj / Spec F dim $= n-1$

  $(F = E(G, X))$

- X_U : smooth for U : suff. small
- $X_U(\mathbb{C}) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}^\infty)/U)$

- $Sh(G,X)$ is a special case of PEL-Shimura

  var.

  X_U has moduli interpretation.

an interpretation : moduli of (A, λ, i, η)

A : abelian scheme

λ : P $\lambda : A \to A^\vee$

• i : E $i : B \to \mathrm{End}(A) \otimes \mathbb{Q}$ Tate module $\otimes \mathbb{Q}$

• η : L an eq. class of $V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\simeq} VA_s$

(or $V \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty} \xrightarrow{\simeq} V^pA_s$ if $\mathbb{Z}_p^\times \subset U$)

§3.3 Statement

ℓ : a prime $\neq p$

ξ : finite dim'l irred. (alg) rep'n of $G \times \operatorname{Spec} \overline{\mathbb{Q}}_\ell$

$\rightsquigarrow \mathcal{L}_\xi$: smooth $\overline{\mathbb{Q}}_\ell$ - sheaf on X_U

$$[H(X, \mathcal{L}_\xi)] := \sum_{i \in \mathbb{Z}} (-1)^i \varinjlim_{U} H^i(X_U \times \operatorname{Spec} \overline{F}, \mathcal{L}_\xi)$$

$$\in \operatorname{Groth}\left(G(\mathbb{A}^\infty) \times \operatorname{Gal}(\overline{F}/F)\right)$$

<u>Main result in [HT]</u>

$$n[H(X, \mathcal{L}_\xi)^{\mathbb{Z}_p^\times}]$$

$$= \sum_{0 \le h \le n-1} \sum_{\rho : \operatorname{Irr}_{\overline{\mathbb{Q}}_\ell}(D^\times_{F_w, n-h})} \operatorname{Ind}_{P_h(F_w)}^{GL_n(F_w)} \begin{pmatrix} \operatorname{Red}^{(h)}_\rho [H(X, \mathcal{L}_\xi)^{\mathbb{Z}_p^\times}] \\ * [\Psi_{F_w, \ell, n-h}(\rho)] \end{pmatrix}$$

in $\operatorname{Groth}(G(\mathbb{A}^\infty) \times W_{F_w})$

where $\cdot (-)^{\mathbb{Z}_p^\times}$ $\mathbb{Z}_p^\times$ - inv. $\mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times = \mathbb{G}_m(\mathbb{Q}_p) \subset G(\mathbb{Q}_p) \subset G(\mathbb{A}^\infty)$

$\cdot$ $D_{F_w, n-h}$: the central div. alg. $/F_w$

s.t.

- $\dim_{F_w} D_{F_w, n-h} = (n-h)^2$
- $\operatorname{inv}[D_{F_w, n-h}] = \frac{1}{n-h} \in \mathbb{Q}/\mathbb{Z}$

$\cdot$ $P_h = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \begin{array}{l} \}\, n-h \\ \}\, h \end{array}$ max parabolic

$$\begin{tikzcd}
\mathrm{red}_\rho^{(h)} : \mathrm{Groth}_{\overline{\mathbb{Q}}_\ell}(GL_n(F_w)) \arrow[rr] \arrow[dr, "①"] & & \mathrm{Groth}_{\overline{\mathbb{Q}}_\ell}(GL_h(F_w)) \\
& \mathrm{Groth}_{\overline{\mathbb{Q}}_\ell}(GL_{n-h}(F_w) \times GL_h(F_w)) \arrow[ur, "②"'] &
\end{tikzcd}$$

where

① "Jacquet-module"

$$[\pi] \longmapsto \left[J_{N_h^{op}}(\pi) \otimes \delta_{P_h}^{1/2} \right]$$

$$N_h^{op} = \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix} \begin{array}{l} \}\, n-h \\ \}\, h \end{array}$$

② "taking $JL(\rho^\vee)$-part.

$$[\alpha \otimes \beta] \longmapsto \begin{cases} \dfrac{1}{\mathrm{vol}(D^\times_{F_w, n-h}/F_w^\times)} \, \mathrm{tr}(\varphi_{JL(\rho^\vee)}; \alpha)\,[\beta] & \text{if } \omega_\alpha = \omega_\rho^{-1} \\ 0 & \text{otherwise} \end{cases}$$

$\varphi_{JL(\rho^\vee)} \in C_c^\infty(GL_{n-h}(F_w), \omega^{-1})$

a pseudo coeff for $JL(\rho^\vee)$

i.e. a fuction satisfying

$$\frac{\mathrm{tr}(\varphi_{\pi(\rho^\vee)}; \alpha)}{\mathrm{vol}(D^\times_{F_w, n-h}/F_w^\times)} = \begin{cases} 1 & \text{if } \alpha = JL(\rho^\vee) \\ & \text{if } \alpha = n\text{-}\mathrm{Ind}_{P(F_w)}^{GL_{n-h}(F_w)} \pi_1 \times \cdots \times \pi_r \\ 0 & \pi_i : \text{irred sq int.} \\ & \text{and } \omega_\alpha = \omega_\rho^{-1}, \alpha \neq JL(\rho^\vee) \end{cases}$$

$\gamma \in GL_{n-h}(F_v)$

non ellip reg s.s.

$= O_\gamma^{GL_h(F_v)}(\varphi_{JL(\rho^\vee}) = 0$

$\gamma \in GL_{n-h}(F_w)$ ellip. reg s.s. $\longrightarrow \delta \in D_{F_w, n-h}$

$\Rightarrow O_\gamma^{GL_{n-h}(F_w)}(\varphi_{JL(\rho^\vee)})$

$= (-1)^{n-h-1} \mathrm{vol}(D^\times_{F_w, n-h}/F_w^\times)$

$\mathrm{tr}(\delta, \rho)$

$\overset{*}{|\mathrm{Art}_{F_w}^{-1}|}$: $\mathrm{Groth}_{\overline{\mathbb{Q}}_\ell}(GL_h(F_w)) \times \mathrm{Groth}_{\overline{\mathbb{Q}}_\ell}(GL_{n-h}(F_w) \times W_F)$

$\mathrm{Art}_{F_w}\ F_w^\times \cong W_{F_w}^{ab}$

$\longrightarrow \mathrm{Groth}_{\overline{\mathbb{Q}}_\ell}(GL_h(F_w) \times GL_{n-h}(F_w) \times W_{F_w})$

$(([\pi_1], [\pi_2 \otimes \sigma]) \longmapsto \pi_1 * (\pi_2 \otimes \sigma)$

as $\overline{\mathbb{Q}}_\ell$-vector space

$(g_1, (g_2, \tau))$ acts ..

as $\pi_1(g_1 |\mathrm{Art}|_{F_w}^{-1}|(\tau)) \otimes \pi_2(g_2) \otimes \sigma(\tau)$

$*\left[\Psi_{F_w,\ell,n-h}(\rho)\right] \in \mathrm{Groth}_{\overline{\mathbb{Q}}_\ell}(GL_{n-h}(F_w)\times W_{F_w})$ (10)

$\wr$

$\Psi^i_{F_w,\ell,n-h,m} := R^i\Psi_\eta(\overline{\mathbb{Q}}_\ell)_{\mathrm{Spf}\, R_{F_w,n-h,m}}$

nearby cycle sheaf (Berkovich)

$\Sigma_{F_w,n-h}$ the 1-dim'l formal $\mathcal{O}_{F_w}$-mod$/\mathrm{Spec}\,\overline{k(w)}$, $ht = n-h$

$\alpha : (\varpi^{-m}/\mathcal{O}_{F_w})^{n-h} \longrightarrow H[\varpi^m](\mathrm{Spec}\, A)$

Drinfeld ϖ^m-level str.

§4. The main result in [HT] – sketch of proof

<u>Step 1</u> Drinfeld level str.

$\rightsquigarrow$ integral model of X_U over $\mathcal{O}_{F_w}$

Step 1-1

Vanishing cycle spectral seq. can express

$[H(X,\mathcal{L}_\xi)^{\mathbb{Z}_p'}]$ in terms of certain

sheaves on $\overline{X}_U = X_U \bmod w$

Step 1-2

(Newton polygon) stratification of $\overline{X_U}$

$$\underbrace{\overline{X_{U^p,m}}}_{\varphi} = \coprod_{0 \le h \le n-1} \overline{X}_{U^p,m}^{(h)}$$

U^p outside p "full level m at w/p"

$$[H(X, \mathcal{L}_\xi)^{\mathbb{Z}_p^\times}] \overset{1.1}{=} (H(\overline{X}, \Phi \otimes \mathcal{L}_\xi))]$$
$$\overset{1.2}{=} \sum_h (H_c(\overline{X}^{(h)}, \Phi \otimes \mathcal{L}_\xi))]$$

Step 1-3

Describe $\overline{X}_{U,m}^{(h)}$ in terms of Igusa var.

$$I_{U^p,m}^{(h)} / \overline{X}_{U^p,\bar{m}}^{(h)} \qquad \underset{\substack{\text{drop level str.} \\ \text{at } w}}{m \longrightarrow \bar{m}}$$

φ moduli space of level str w^{m_w} on $\mathcal{G}^{ét}$ where $\mathcal{G} \leftarrow A[\varpi^\infty]^{univ}$

$\exists \; (I_{U^p,m}^{(h)})^\wedge / (X_{U^p,\bar{m}}^{(h)})^\wedge$: natural formal ext'n.

$$(I_{U^p,m}^{(h)})^\wedge(t) / (I_{U^p,m}^{(h)})^\wedge \quad (t \geqq 0)$$

of Drinfeld w-str. on $\mathcal{G}^0$.

Relate $\coprod_{M \subset (\mathcal{O}_{F_w}/\varpi^{m_w})^{\oplus n}}$ direct summand of rk $n-h$

$$\rightsquigarrow [H(X,\mathcal{L}_\xi)^{\mathbb{Z}_p^\times}] = \sum_{0\le h\le n-1} \mathrm{Ind}_{P_h(F_w)}^{GL_n(F_w)} [H_c(I^{(h)}, \Phi\otimes\mathcal{L}_\xi)]$$

<u>Step 2</u>

$J^{(h)}_{U^p,m,s}$ Igusa var. of 2nd kind.

$s \in \mathbb{Z}_{\ge 0}$

$$J^{(h)}_{U^p,m,s} \longrightarrow I^{(h)}_{U^p,m} \times \operatorname{Spec} \overline{k(w)}$$

Galois covering with

Galois gp $(\mathcal{O}_{D_{F_w,n-h}}/w^s)^\times$

$$\mathcal{O}_{D_{F_w,n-h}} \curvearrowright \Psi^j_{F_w,l,n-h,t}$$

$\rightsquigarrow \mathcal{F}(\Psi^j_{F_w,l,n-h,t})$ smooth $\overline{\mathbb{Q}}_l$ - sheaf on $I^{(h)}_{U^p,m} \times \operatorname{Spec} \overline{k(w)}$

<u>Key point</u>

on $I^{(h)}_{U^p,m} \times \operatorname{Spec} \overline{k(w)}$

$\exists$ two $\overline{\mathbb{Q}}_l$ - sheaves : $\Phi^j(t)$ & $\mathcal{F}(\Psi^j_{F_w,l,n-h,t})$

$\exists$ can isom $\kappa : \Phi^j(t) \xrightarrow{\simeq} \mathcal{F}(\Psi^j_{F_w,l,n-h,t})$

$$H^i_c(I^{(h)}, \Phi^j\otimes\mathcal{L}_\xi) \cong H^i_c(I^{(h)}, \mathcal{F}(\Psi^j)\otimes\mathcal{L}_\xi)$$

decompose $\mathcal{F}(\Psi^j) = \bigoplus_\rho \mathcal{F}_\rho \boxtimes \Psi^j(\rho)$

$$\rightsquigarrow \left[H(X,\mathcal{L}_\xi)^{\mathbb{Z}_p^\times}\right] = \sum_{0\le h\le n-1} \sum_{\rho} e(\rho)^{-1} \mathrm{Ind}_{P_h(F_w)}^{GL_n(F_w)} (-)$$

$$(-) = \left[H_c^{\cdot}(I^{(h)}, \mathcal{F}_\rho \otimes \mathcal{L}_\xi)\right] \underset{d_h}{*} \left[\Psi_{F_w,l,n-h}(\rho)\right]$$

in $\mathrm{Groth}_{\overline{\mathbb{Q}}_l}(G(\mathbb{A}^\infty)\times W_{F_v})$

(1st basic identity)

$\rho \in \mathrm{Irr}_{\overline{\mathbb{Q}}_l}(D_{F_w,n-h}^\times)$

$e[\rho] = \mathrm{length}\ \rho|_{\mathcal{O}_{D_{F_w,n-h}}}$

$d_h : GL_{n-h}(F_w)\times W_{F_v} \longrightarrow \mathbb{Q}_p^\times$

$(\gamma,\sigma) \longmapsto \left|\dfrac{\det\gamma}{\mathrm{Art}_{F_w}^{-1}(\sigma)}\right|$

Step 3

Step 3-1

Apply Fujiwara's trace formula

$\varphi \in C_c^\infty\left(G^{(h)}(\mathbb{A}^\infty)^+ / \mathbb{Z}_p^\times \times \mathcal{O}_{D_{F_w,n-h}}^\times\right)$

$$G^{(h)}(\mathbb{A}^\infty)^+ = G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times \left(D_{F_w,n-h}^\times \times GL(F_w)\right)^+ \times \prod_{\substack{w'|u \\ w'\ne w}} (B_{w'}^{op})^\times$$

$$\mathrm{tr}\,(\varphi : H_c(I^{(h)}, \mathcal{F}_\rho \otimes \mathcal{L}_\xi)$$

$$= \sum_{z \in PIC^{(h)}} \sum_{a \in H_z(\mathbb{Q})} \begin{pmatrix} \text{some} \\ \text{vol} \end{pmatrix} \begin{pmatrix} \text{some} \\ \text{orbital} \\ \text{integral} \end{pmatrix} \mathrm{tr}\,(\rho \otimes \xi(\eta_z(a))$$

where

$PIC^{(h)}$ "polarized isog. class"

$\parallel$

$\{(A, \lambda, i)\}/\sim$ isogeny

$$\begin{pmatrix} A : \text{abl. var}/\overline{k(v)} \quad \dim = dn^2 \\ + \text{conditions} \end{pmatrix}$$

for $z = [(A, \lambda, i)] \in PIC^{(h)}$

H_z alg gp/$\mathbb{Q}$

$$H_z = \{(\lambda, g) \in \mathbb{G}_m \times End(A) \otimes \mathbb{Q},\ g g^{\#} = \lambda\}$$

$\rightsquigarrow \eta_z : H_z(\mathbb{A}^\infty) \hookrightarrow G^{(h)}(\mathbb{A}^\infty)$

can. up to $G^{(h)}(\mathbb{A}^\infty)$-conjugacy

Step 3-2

Use Honda-Tate theory

$\xrightarrow[\text{Kottwitz techniques}]{}$ describe $PIC^{(h)}$, $H_z(\mathbb{Q})$

in terms of $FP_\tau^{(h)}$, where

$$FP_\tau^{(h)} = \left\{ (a, \tilde{w}) \;\middle|\; \begin{array}{l} a \in G(\mathbb{Q}) \ (\subset B) \\ a : \text{ellip. in } G_\tau(\mathbb{R}) \\ \tilde{w} : \text{a place of } F(a) \subset B \text{ s.t. } \tilde{w}/w \end{array} \right.$$

$$(n-h)[F(a):F] = n[F(a)_{\tilde{w}} : F_w] \quad \Big) /_n$$ (15)

$$\Rightarrow \quad tr(\varphi ; H_c(I^{(h)}, \mathcal{F}_\rho \otimes \mathcal{L}_\xi))$$

$$= \underset{\substack{\| \\ 1 \text{ or } 2}}{(-1)^h \kappa_B} \sum_{y=(a,\tilde{w})} \frac{(\text{some orb integral})\ tr((\rho\otimes\xi)(\iota(y)))}{[F(a):F]\ (\text{volume})}$$

<u>Step 3-3</u>

calculation

$$\varphi = \varphi^w \times \varphi^o \times \varphi^e$$

$$\varphi^o \in C_c^\infty(D_{F_w, n-h}^\times / \mathcal{O}_{D_{F_w, n-h}}^\times)$$

$$\varphi^e \in C_c^\infty(GL_h(F_w))$$

$$n \cdot tr(\varphi ; H_c(I^{(h)}, \mathcal{F}_\rho \otimes \mathcal{L}_\xi))$$

$$= n\ (\text{RHS of the formula in Step 3-2})$$

$$= (-1)^n \kappa_B \cdot n \cdot \sum_a \frac{\text{const}}{\text{vol}} O_a(\varphi^w \times IPC_\rho(\varphi^o, \varphi^e, \mu_n, \mu_h))$$

$$= tr(\varphi^w \times IPC_\rho ; H(X, \mathcal{L}_\xi))$$

$$= tr(\varphi ; Red_\varphi^{(h)} H(X, \mathcal{L}_\xi))$$

By defn. of $IPC_\rho(-) \in C_c^\infty(\quad)$

2nd basic identity

$$\begin{tikzcd}[column sep=small] \mathrm{Red}_\rho^{(h)} : \mathrm{Groth}(GL_n(F_w)) \arrow[dr, "①"'] & & \mathrm{Groth}(D^\times_{F_w, n-h}/\mathcal{O}^\times_{D_{F_w, n-h}} \times GL_h(F_w)) \\ & \mathrm{Groth}(GL_{n-h}(F_w) \times GL_h(F_w)) \arrow[ur, "②"'] & \end{tikzcd}$$

① same as $\mathrm{red}_\rho^{(h)}$

② $[\alpha \otimes \beta] \mapsto \sum \dfrac{\mathrm{tr}(\varphi_{JL(\rho^\vee)}; \alpha)}{\mathrm{vol}(D/F_w^\times)} [\psi \otimes \beta]$

ψ : char. of $D^\times/\mathcal{O}_D^\times$ s.t. $\omega_\alpha = \psi \omega_\rho^{-1}$

1st + 2nd basic identity

$\Rightarrow$ Main result in [HT]